

ANIMAL BREEDING NOTES**CHAPTER 1****DEFINITIONS AND ELEMENTARY MATRIX OPERATIONS****Definitions**

Matrix: set of numbers arranged by rows and columns. For example, suppose that three bulls have the following numbers of progeny in two herds:

Herds	Bulls		
	1	2	3
1	4	6	1
2	5	8	3

These numbers of progeny per bull can define a rectangular matrix A with 2 rows and 3 columns:

$$A_{2 \times 3} = \begin{bmatrix} 4 & 6 & 1 \\ 5 & 8 & 3 \end{bmatrix}$$

The dimension of A is defined in terms of its number of rows and columns. Thus, the matrix A has dimension 2×3 . Each number in $A_{2 \times 3}$ is called an entry or element.

In general,

$$A_{r \times c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} = \{a_{ij}\}_{r \times c}$$

where $a_{ij} = ij^{\text{th}}$ element of $A_{r \times c}$.

Square matrix: matrix where the number of rows (r) is equal to number of columns (c). For instance,

$$B_{2 \times 2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B_2 = \{b_{ij}\}_2 \text{ is a square matrix.}$$

The following are square matrices that have special structures:

a) **Diagonal matrix:** offdiagonal elements are zero:

$$D_2 = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} = \text{diag } \{d_{ii}\}_2$$

b) **Identity matrix:** offdiagonal elements are zero and diagonal elements are 1's.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c) **Upper (lower) triangular matrix:** elements below (above) the diagonal are zero:

$$U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$L = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

Vector: array of numbers forming a single column (column vector) or a single row (row vector).

For example:

[1-3]

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ is a column vector and,}$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \text{ is a row vector.}$$

Scalar: a single number, e.g., 5. A scalar can be considered to be 1×1 matrix.

Summation notation:

$$\sum_{i=1}^n v_i = v_1 + v_2 + v_3 + \dots + v_n$$

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n a_{ij} &= a_{11} + a_{12} + \dots + a_{1n} + a_{21} + a_{22} + \dots + a_{2n} + \dots + a_{m1} + a_{m2} + \\ &\dots + a_{mn} \end{aligned}$$

$$\sum_{k=1}^m a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{im} b_{mj}$$

Transpose of a matrix: matrix obtained by interchanging the rows of the original matrix. Thus, the transpose of $A_{r \times c}$, denoted as $A'_{c \times r}$, is:

$$A'_{c \times r} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{r1} \\ a_{12} & a_{22} & \dots & a_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1c} & a_{2c} & \dots & a_{rc} \end{bmatrix}$$

Symmetric matrix (square matrix only): An arbitrary matrix A is said to be symmetric if $A =$

A' . Thus,

$$A \text{ symmetric} \Rightarrow \{a_{ij} = a_{ji}\}.$$

Examples of symmetric matrices are:

$$D_2 = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}.$$

Matrix operations

Addition: Defined only for matrices of the same dimensions,

$$A_{r \times c} + B_{r \times c} = \{a_{ij} + b_{ij}\}_{r \times c}$$

Subtraction: for $A_{r \times c}$ and $B_{r \times c}$

$$A - B = \{a_{ij} - b_{ij}\}$$

Scalar multiplication: for an arbitrary scalar c and matrix $A_{r \times c}$,

$$cA = \{ca_{ij}\} = \{b_{ij}\}$$

Results: for matrices A, B and C of equal dimension and scalars c and d,

a) $(A + B) + C = A + (B + C),$

b) $A + B = B + A,$

c) $c(A + B) = cA + cB,$

d) $(c + d) A = cA + dA,$

e) $(A + B)' = A' + B',$

f) $(cd)A = c(dA),$ and

g) $(cA)' = cA'.$

Matrix multiplication: The product of matrix $A_{u \times v}$ times matrix $B_{m \times n}$ exists if and only if the number of columns of A (i.e., v) is equal to the number of rows of B (i.e., m). If so, the resulting matrix C is an $u \times n$ matrix whose ij^{th} element (c_{ij}) is equal to the sum of the products of the elements in the i^{th} row of A by the elements in the j^{th} column of B, i.e.,

$$C = AB \text{ exists if } v = m, \text{ and } \{c_{ij}\} = \left(\sum_{k=1}^m a_{ik} b_{kj} \right)$$

For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

Numerically,

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 2 \\ 6 & 6 & 1 \end{bmatrix}$$

Remarks:

a) $AA = A^2$ exists only for square matrices.

b) If AB exists it does **not** imply that BA exists, unless $\dim A = \text{reciprocal of } \dim B$, e.g., $A_{3 \times 4}$

and $B_{4 \times 3}$. A special case is the multiplication of a matrix by its transpose, i.e.,

AA' and $A'A$ always exists.

c) if $A = \text{square and symmetric}$, then

$$A'A = AA' = A^2.$$

Results: for matrices A , B and C of appropriate dimensions and a scalar c ,

a) $c(AB) = (cA)B$,

b) $A(BC) = (AB)C$,

c) $A(B + C) = AB + AC$,

d) $(B + C)A = BA + CA$, and

e) $(AB)' = B'A'$.

Vector multiplication: for matrices $A_{r \times c}$ and $B_{c \times n}$ and vectors $y_{c \times 1}$ and $x_{c \times 1}$

a) $y_{1 \times c}' x_{c \times 1} = p_{1 \times 1}$ is a scalar,

b) $x_{c \times 1} y_{1 \times c}' = P_{c \times c}$ is a matrix,

c) $A_{r \times c} x_{c \times 1} = p_{r \times 1}$ is a column vector, and

d) $y_{1 \times c}' B_{c \times n} = p_{1 \times n}'$ is a row vector.

Examples:

a) $\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 11$

b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 8 & 2 \\ 2 & 4 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}$

Trace of a matrix (square matrix only): sum of the diagonal elements. For an arbitrary square matrix A_n ,

$$\text{tr } A_n = \sum_{i=1}^n a_{ii}$$

Note: If $A_n = I_n$, then $\text{tr } I_n = n$.

Results: for matrices A and B of appropriate dimensions and a scalar c,

a) $\text{tr } (A + B) = \text{tr } (A) + \text{tr } (B),$

b) $\text{tr } (A') = \text{tr } (A),$

c) $\text{tr}(cA) = c \text{tr } (A),$

d) $\text{tr}(AB) = \text{tr } (BA) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}, \text{ and}$

e) $\text{tr } (AA') = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$ if A is a square matrix.

Partitioned matrices

Frequently matrices are handled more easily in a partitioned form. For instance, $A_{m \times n}$ can be partitioned as:

$$A = [A_1 \mid A_2]$$

where $A_1 = m \times n_1$, $A_2 = m \times n_2$, and $n_1 + n_2 = n$.

Example:

$$A = [A_1 | A_2] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array} \right]$$

Finer partitioning of matrices are also possible. Thus, A could be partitioned into four submatrices:

$$A = \left[\begin{array}{cc|cc} A_{11} & & A_{12} & \\ A_{21} & & A_{22} & \end{array} \right] \quad \text{and} \quad A' = \left[\begin{array}{cc|cc} A_{11}' & & A_{21}' & \\ A_{12}' & & A_{22}' & \end{array} \right]$$

Example:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & \\ -- & -- & -- & \\ 0 & 1 & 2 & \end{array} \right] \quad \text{and} \quad A' = \left[\begin{array}{cc|cc} 1 & & 0 & \\ 2 & & 1 & \\ -- & & -- & \\ 3 & & 2 & \end{array} \right]$$

Partitioned matrix operations

Addition and Subtraction: submatrices must be of the same order.

$$A + B = \left[\begin{array}{cc|cc} A_{11} + B_{11} & & A_{12} + B_{12} & \\ A_{21} + B_{21} & & A_{22} + B_{22} & \end{array} \right]$$

Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -- & -- & -- \\ 0 & 1 & 2 \end{array} \right] + \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -- & -- & -- \\ 1 & 1 & 1 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 3 & 3 \\ -- & -- & -- \\ 1 & 2 & 3 \end{array} \right]$$

Multiplication: submatrices must be conformable for multiplication. For example, let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Then, the product AB in partitioned form is:

$$AB = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ \text{-----} \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ \text{---} \\ C_2 \end{bmatrix}$$

The product AB exists in partitioned form if $A_{11}B_1$, $A_{12}B_2$, $A_{21}B_1$ and $A_{22}B_2$ exist and if $A_{11}B_1$ and $A_{12}B_2$ as well as $A_{21}B_1$ and $A_{22}B_2$ are conformable for addition. Thus,

number of columns of A_{11} and A_{21} = number of rows of B_1 , and

number of columns of A_{12} and A_{22} = number of rows of B_2 .

Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -- & -- & -- \\ 0 & 1 & 2 \end{array} \right] \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ \text{---} \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{c} [1 \ 2] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + [3][0 \ 1] \\ \text{-----} \\ [0 \ 1] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + [2][0 \ 1] \end{array} \right]$$

$$= \begin{bmatrix} [2 \ 3] + [0 \ 3] \\ \hline [1 \ 1] + [0 \ 2] \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ \hline 1 & 3 \end{bmatrix}$$

Partitioned matrices can be multiplied as if the submatrices were real numbers, provided that the partition of the columns of the first matrix is the same as the partition of the rows of the second matrix. Thus, the product of any two conformable matrices A and B in partitioned form is:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1u} \\ A_{21} & A_{22} & \cdots & A_{2u} \\ \vdots & \vdots & \vdots & \vdots \\ A_{v1} & A_{v2} & \cdots & A_{vu} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1w} \\ B_{21} & B_{22} & \cdots & B_{2w} \\ \vdots & \vdots & \vdots & \vdots \\ B_{u1} & B_{u2} & \cdots & B_{uw} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1w} \\ C_{21} & C_{22} & \cdots & C_{2w} \\ \vdots & \vdots & \vdots & \vdots \\ C_{v1} & C_{v2} & \cdots & C_{vw} \end{bmatrix}$$

where $C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$.

Direct sum: defined for matrices of any order. The direct sum of matrices $A_{m \times n}$ and $B_{r \times c}$ is:

$$A \oplus B = \begin{bmatrix} A & 0_{12} \\ 0_{21} & B \end{bmatrix} = C_{m+r, n+c} \text{ (block-diagonal),}$$

where,

$0_{12} = m \times c$ null matrix, and

$0_{21} = r \times n$ null matrix.

Example:

$$\begin{bmatrix} 3 & 0 & 4 \\ 1 & 5 & 6 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left[\begin{array}{ccc|c} 3 & 0 & 4 & 0 \\ 1 & 5 & 6 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array} \right]$$

Results:

1) $A \oplus (-A) = 0$ if $A = 0$.

2) $(A \oplus B) + (C \oplus D) = (A + C) \oplus (B + D)$ if conformability for addition is satisfied, i.e., order

$A = \text{order } C$ and $\text{order } B = \text{order } D$.

3) $(A \oplus B)(C \oplus D) = AC \oplus BD$ if matrices are conformable for multiplication, i.e., the number

of columns of A equals the number of rows of B and, the number of columns of B is the same as the number of rows of D .

Direct product: also called Kronecker product. Matrices can be of any order. The direct product of matrices $A_{m \times n}$ and $B_{r \times c}$ is:

$$A * B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} = C_{mr \times nc}$$

Example:

$$\begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix} * \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 12 & | & 2 & 3 \\ \text{-----} & | & \text{-----} \\ 4 & 6 & | & 12 & 18 \end{bmatrix}$$

Results:

1) for any matrixes A and B

$$(A * B)' = A' * B'$$

2) for vectors u and v

$$u' * v = v * u' = vu'$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} * \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

3) for a diagonal matrix of order r and a matrix A,

$$D * A = d_1 A \oplus d_2 A \oplus \dots \oplus d_r A$$

4) for partitioned matrices A and B,

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} * B = \begin{bmatrix} A_1 * B & A_2 * B \end{bmatrix}$$

$$\begin{bmatrix} 2 & | & 2 \\ 1 & | & 3 \end{bmatrix} * \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} * \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \mid \begin{bmatrix} 2 \\ 3 \end{bmatrix} * \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix} \right]$$

$$\left[\begin{array}{cc|cc} 8 & 2 & 8 & 2 \\ 0 & 4 & 0 & 4 \\ 4 & 1 & 12 & 3 \\ 0 & 2 & 0 & 6 \end{array} \right] = \left[\begin{array}{cc|cc} 8 & 2 & 8 & 2 \\ 0 & 4 & 0 & 4 \\ 4 & 1 & 12 & 3 \\ 0 & 2 & 0 & 6 \end{array} \right]$$

5) for partitioned matrices A and B,

$$A * [B_1 \quad B_2] \neq [A * B_1 \quad A * B_2]$$

$$\left[\begin{array}{cc} 2 & 2 \\ 1 & 3 \end{array} \right] * \left[\begin{array}{c|c} 4 & 1 \\ 0 & 2 \end{array} \right] \neq \left[\left[\begin{array}{cc} 2 & 2 \\ 1 & 3 \end{array} \right] * \left[\begin{array}{c} 4 \\ 0 \end{array} \right] \mid \left[\begin{array}{cc} 2 & 2 \\ 1 & 3 \end{array} \right] * \left[\begin{array}{c} 1 \\ 2 \end{array} \right] \right]$$

$$\left[\begin{array}{cc|cc} 8 & 2 & 8 & 2 \\ 0 & 4 & 0 & 4 \\ \hline 4 & 1 & 12 & 3 \\ 0 & 2 & 0 & 6 \end{array} \right] \neq \left[\begin{array}{cc|cc} 8 & 8 & 2 & 2 \\ 0 & 0 & 4 & 4 \\ \hline 4 & 12 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{array} \right]$$

6) If matrices A and B as well as C and D are conformable for multiplication,

$$(A * B)(C * D) = (AC * BD)$$

$$\left(\left[\begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array} \right] * [2] \right) \left(\left[\begin{array}{c} 4 \\ 2 \end{array} \right] * [1] \right) = \left(\left[\begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array} \right] \left[\begin{array}{c} 4 \\ 2 \end{array} \right] * [2][1] \right)$$

$$\left[\begin{array}{cc} 4 & 2 \\ 2 & 6 \end{array} \right] \left[\begin{array}{c} 4 \\ 2 \end{array} \right] = \left[\begin{array}{c} 10 \\ 10 \end{array} \right] * [2]$$

$$\left[\begin{array}{c} 20 \\ 20 \end{array} \right] = \left[\begin{array}{c} 20 \\ 20 \end{array} \right]$$

Determinants

Scalar function (polynomial) of the elements of a square matrix. The determinant of a rectangular matrix is undefined and does not exist.

The determinant of a matrix $A_{n \times n}$ is called an n-order determinant, and it is written as:

$$\det A = |A|$$

Evaluating (or reducing or expanding) a determinant

The determinant of the (square) matrix $A_{n \times n}$ is:

$$|A| = a_{11} \quad \text{if } n = 1$$

$$|A| = \sum_{j=1}^n a_{ij} |A_{ij}| (-1)^{i+j} \quad \text{for any row } i \quad \text{if } n > 1, \text{ or}$$

$$= \sum_{i=1}^n a_{ij} |A_{ij}| (-1)^{i+j} \quad \text{for any column } j \quad \text{if } n > 1$$

where

$A_{ij} = (n-1) \times (n-1)$ submatrix of A obtained by deleting the i^{th} row and the j^{th} column of A ,

$|A_{ij}| \equiv$ minor of element a_{ij} in $|A|$,

$|A_{ij}| (-1)^{i+j} \equiv$ cofactor of element a_{ij} in $|A|$, i.e., it is a signed minor.

Examples:

1) 1×1 matrix

$$|3| = 3$$

2) 2×2 matrix

$$\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = (4)(3) - (2)(1)$$

$$= 10$$

3) 3×3 matrix

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} (-1)^{2+1} + 4 \begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} (-1)^{2+2} + 0 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} (-1)^{2+3}$$

$$= -2(12 - 0) + 4(18 - 1) + 0$$

$$= 44$$

This computational method is called expansion by the elements of a row (or a column) or expansion by minors.

Remarks:

- 1) Interchanging two rows or two columns of a determinant changes its sign.
- 2) If a scalar is a factor of a row it is also a factor of the determinant.
- 3) If a row of a determinant is a multiple of another row, the determinant is zero.

4) Adding a multiple of a row to another row of a determinant does not change its value.

5) The determinant of λA , where λ is a scalar and A an $n \times n$ matrix is $\lambda^n |A|$, e.g.,

$$|-A| = (-1)^n |A|$$

6) for $A_{n \times n}$

$$|A'| = |A|$$

7) for $A_{n \times n}$ and $B_{n \times n}$

$$|AB| = |A| |B|$$

8) if $A_{n \times n}$ is diagonal

$$|A| = a_{11} a_{22} \dots a_{nn}$$

9) for $A_{n \times n}$, define:

$$A \equiv \text{singular if } |A| = 0$$

$$A \equiv \text{nonsingular if } |A| \neq 0$$

References

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