

## ANIMAL BREEDING NOTES

### CHAPTER 11

#### THE MIXED LINEAR MODEL

##### **Definition of the mixed linear model**

The mixed linear model enormously facilitates the computation of:

- 1) The Generalized Least Squares (GLS) of fixed effects, and
- 2) The Best Linear Unbiased Predictors (BLUP) of random effects, particularly in large unbalanced data sets.

The usual representation of the mixed linear model is:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Zg} + \mathbf{e}$$

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{g} \\ \mathbf{e} \end{bmatrix} \sim \left\{ \begin{bmatrix} \mathbf{X}\boldsymbol{\beta} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{ZGZ}' + \mathbf{R} & | & \mathbf{ZG} & \mathbf{R} \\ \hline & | & \mathbf{GZ}' & | & \mathbf{G} & \mathbf{0} \\ & & \mathbf{R} & | & \mathbf{0} & \mathbf{R} \end{bmatrix} \right\}$$

where

$\mathbf{y} = [\mathbf{y}_1 \mathbf{y}_2 \dots \mathbf{y}_n]',$  vector of observations,

$\boldsymbol{\beta} = [\beta_1 \beta_2 \dots \beta_l]',$  vector of fixed effects,

$\mathbf{g} = [\mathbf{g}_1 \mathbf{g}_2 \dots \mathbf{g}_l]',$  vector of random genetic effects,

$\mathbf{e} = [\mathbf{e}_1 \mathbf{e}_2 \dots \mathbf{e}_n]',$  vector of random residual effects,

$\mathbf{X} = [\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_l],$  incidence matrix relating elements of  $\mathbf{y}$  to elements of  $\boldsymbol{\beta},$

$\mathbf{Z} = [\mathbf{Z}_1 \mathbf{Z}_2 \dots \mathbf{Z}_J],$  incidence matrix relating elements of  $\mathbf{y}$  to elements of  $\mathbf{g}.$

The BLUP of  $\mathbf{g}$  (Chapter 10) is:

[11-2]

$$\hat{g} = C'V^{-1}(y - X\beta^\circ)$$

Let

$$V = ZGZ' + R$$

$$C' = GZ'$$

Thus, in terms of the mixed linear model, the BLUP of  $g$  is:

$$\hat{g} = GZ'(ZGZ' + R)^{-1}(y - X\beta^\circ).$$

Also, the BLUP of  $e$  is:

$$\hat{e} = \text{cov}(e, y')[\text{var}(y)]^{-1}[y - X\beta^\circ]$$

$$\hat{e} = RV^{-1}(y - X\beta^\circ).$$

**Note:** to prove that the BLUP of  $e$  is equal to  $RV^{-1}(y - X\beta^\circ)$  use the minimization procedure described in Chapter 10. Let  $w = K'\beta + e$ , consider the joint distribution of  $y$  and  $e$ , i.e.,

$$\begin{bmatrix} y \\ e \end{bmatrix} = \left\{ \begin{bmatrix} X\beta \\ V & R \\ 0 \\ R & R \end{bmatrix}, \begin{bmatrix} V & R \\ R & R \end{bmatrix} \right\}, \text{ and minimize } E[(a + By - K'\beta - e)'A(a + By - K'\beta - e)] +$$

$\text{tr}(2M'(BX - K'))$  with respect to  $a$ ,  $M'$ , and  $B$ . The result will be  $\hat{w} = K'\beta^\circ + \hat{e}$  = BLUP of  $w$ , where  $\hat{e} = RV^{-1}(y - X\beta^\circ)$ .

In terms of the mixed linear model the BLUP of  $e$  is:

$$\hat{e} = R(ZGZ' + R)^{-1}(y - X\beta^\circ)$$

$$\text{But } R = V - ZGZ'.$$

Thus,

$$\hat{e} = (V - ZGZ')V^{-1}(y - X\beta^\circ)$$

$$\hat{e} = y - X\beta^\circ - ZGZ'V^{-1}(y - X\beta^\circ)$$

$$\Rightarrow \hat{e} = y - X\beta^\circ - Z\hat{g}, \text{ because } \hat{g} = GZ'V^{-1}(y - X\beta^\circ).$$

[11-3]

The **disadvantage** of the BLUP equations for  $\mathbf{g}$  and  $\mathbf{e}$  above, is the requirement to invert  $\mathbf{V} = \mathbf{ZGZ}' + \mathbf{R}$ , which is often large and nondiagonal. Thus,  $\mathbf{V}^{-1}$  may be impossible or too expensive to compute under the current computer capabilities. However, if  $\text{cov}(\mathbf{g}, \mathbf{e}') = \mathbf{0}$ , then

$$\begin{aligned}\mathbf{V}^{-1} &= (\mathbf{ZGZ}' + \mathbf{R})^{-1} \\ \mathbf{V}^{-1} &= \mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{Z}(\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}.\end{aligned}$$

The expression  $\mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{Z}(\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}$  is **easier** to compute than  $\mathbf{V}^{-1}$  itself, because **R** and **G** usually have patterns, so their **inverses are easy to compute**. For instance, R may be block-diagonal and G may be equal to  $\mathbf{G}_0 * \mathbf{A}$ , where A is the matrix of additive relationships and  $\mathbf{G}_0$  is a small matrix of additive genetic covariances. Because  $\mathbf{A}^{-1}$  can be computed using simple rules and  $\mathbf{G}_0^{-1}$  can be computed directly,  $\mathbf{G}^{-1} = \mathbf{G}_0^{-1} * \mathbf{A}^{-1}$  is easily obtained. Henderson's mixed model procedures make use of  $\mathbf{V}^{-1} = (\mathbf{ZGZ}' + \mathbf{R})^{-1}$ .

### C. R. Henderson's derivation of BLUP using the mixed model equations

C. R. Henderson came up with his mixed model equations (MME) when trying to find a more feasible procedure to obtain maximum likelihood estimators (MLE) of fixed effects. C. R. Henderson realized that he could obtain the same MLE estimators of  $\beta$  ( $\beta^\circ = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$ ) by solving the set of equations obtained by maximizing the joint density function  $f(\mathbf{y}, \mathbf{g})$ , written as  $f(\mathbf{y}|\mathbf{g})f(\mathbf{g})$ . Notice that C. R. Henderson did not condition on  $\beta$  and  $\mathbf{g}$ , but only on  $\mathbf{g}$ . These same MME can be obtained by maximizing the posterior density of  $f(\beta, \mathbf{g}|\mathbf{y})$ , written as  $f(\mathbf{y}|\beta, \mathbf{g})f(\beta, \mathbf{g})$ , and assuming that  $\sigma_\beta^2 \rightarrow \infty$  so that  $[\text{var}(\beta)]^{-1} = \mathbf{I} \sigma_\beta^{-2} \rightarrow 0$  (Quaas, 1986).

Assume

[11-4]

$$\begin{bmatrix} y \\ g \\ e \end{bmatrix} \sim MVN \left\{ \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ZGZ' + R & | & ZG & R \\ \cdots & | & \cdots & \cdots \\ GZ' & | & G & 0 \\ R & | & 0 & R \end{bmatrix} \right\}.$$

Maximize

$$f(y, g) = f(y|g)f(g)$$

where

$$\begin{aligned} f(y|g) &\sim MVN \{ E[y] + cov(y, g)' var(g)^{-1} (g - E[g]), var(y) - cov(y, g)' var(g)^{-1} cov(g, y') \} \\ &\sim MVN \{ X\beta + ZGG^{-1}(g - 0), \textcolor{red}{ZGZ'} + R - \textcolor{red}{ZGG^{-1}GZ'} \} \\ &\sim MVN \{ X\beta + Zg, R \} \end{aligned}$$

and

$$f(g) \sim MVN \{ 0, G \}.$$

Let

$$L = f(y|g)f(g)$$

$$\begin{aligned} L &= (2\pi)^{-\frac{n}{2}} |R|^{-\frac{J}{2}} \exp \left\{ -\frac{1}{2} (y - X\beta - Zg)' R^{-1} (y - X\beta - Zg) \right\} \\ &\quad \times (2\pi)^{-\frac{J}{2}} |G|^{-\frac{J}{2}} \exp \left\{ -\frac{1}{2} (g - 0)' G^{-1} (g - 0) \right\} \end{aligned}$$

Because  $\log L$  is an increasing function of  $L$ , its maximum is located at the same point in the space of  $[\beta \ g]'$  and  $ZGZ' + R$  as the maximum of  $L$ . Thus, maximizing  $\log L$  is equivalent to maximizing  $L$ . Let  $\Psi = \log L$ . Then,

$$\begin{aligned} \Psi &= -\frac{1}{2} (n+J) \log(2\pi) - \frac{1}{2} (\log|R| + \log|G|) \\ &\quad - \frac{1}{2} (y - X\beta - Zg)' R^{-1} (y - X\beta - Zg) - \frac{1}{2} g' G^{-1} g \\ \Psi &= \text{constants} - \frac{1}{2} (y' R^{-1} y - 2y' R^{-1} X\beta - 2y' R^{-1} Zg) \end{aligned}$$

[11-5]

$$+ \beta' X' R^{-1} X \beta + 2\beta' X' R^{-1} Z g + g' Z' R^{-1} Z g + g' G^{-1} g)$$

$$\frac{\partial \Psi}{\partial \beta} = -X' R^{-1} y + X' R^{-1} X \beta + X' R^{-1} Z g = 0$$

$$\Rightarrow X' R^{-1} X \beta + X' R^{-1} Z g = X' R^{-1} y$$

$$\frac{\partial \Psi}{\partial g} = -Z' R^{-1} y + Z' R^{-1} X \beta + Z' R^{-1} Z g + G^{-1} g = 0$$

$$\Rightarrow Z' R^{-1} X \beta + (Z' R^{-1} Z + G^{-1}) g = Z' R^{-1} y$$

The resulting set of MME is:

$$\begin{bmatrix} X' R^{-1} X & X' R^{-1} Z \\ Z' R^{-1} X & Z' R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ g \end{bmatrix} = \begin{bmatrix} X' R^{-1} y \\ Z' R^{-1} y \end{bmatrix}$$

These are **Henderson's MME**, or simply, the **MME**.

### Theorem.

Given

$$\begin{bmatrix} y \\ \vdots \\ g \\ e \end{bmatrix} \sim MVN \left\{ \begin{bmatrix} X\beta \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ZGZ' + R & ZG & R \\ \cdots & \cdots & \cdots \\ GZ' & G & 0 \\ R & 0 & R \end{bmatrix} \right\}$$

- (a)** The MME yield GLS solutions for the fixed effects ( $\beta^\circ$ ) and BLUP of the random effects ( $\hat{g}$ ), and

$$\textbf{(b)} [K' L'] \begin{bmatrix} X' R^{-1} X & X' R^{-1} Z \\ Z' R^{-1} X & Z' R^{-1} Z + G^{-1} \end{bmatrix}^{-1} \begin{bmatrix} K \\ L \end{bmatrix}$$

is the EVP of the BLUP of  $w = K'\beta + L'g$ , i.e., the EVP of  $\hat{w} = K'\beta^\circ + L'\hat{g}$ .

### Proof:

[11-6]

(a.1)  $\beta^\circ$  from the MME is the GLS of  $\beta$

Absorb  $\hat{g}$  into the  $\beta^\circ$  equations:

$$\begin{aligned} (X'R^{-1}X - X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}X)\beta^\circ &= (X'R^{-1}y - X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}y) \\ X'(R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1})X\beta^\circ &= X'(R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1})y. \end{aligned}$$

But the GLS equations for  $\beta^\circ$  are:

$$X'V^{-1}X\beta^\circ = X'V^{-1}y.$$

$\Rightarrow$  If  $\beta^\circ$  from the MME is the GLS of  $\beta$  we only need to show that:

$$V^{-1} = R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}.$$

Multiplying both sides by  $V = ZGZ' + R$  gives:

$$\begin{aligned} I &= (R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1})(ZGZ' + R) \\ I &= R^{-1}ZGZ' + I - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}ZGZ' - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z' \\ I &= R^{-1}ZGZ' + I - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}(Z'R^{-1}ZGZ' + Z') \\ I &= R^{-1}ZGZ' + I - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}(Z'R^{-1}Z + G^{-1})GZ' \\ I &= R^{-1}ZGZ' + I - R^{-1}ZGZ' \\ I &= I. \end{aligned}$$

$$\Rightarrow V^{-1} = R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}.$$

$\Rightarrow \beta^\circ$  from the MME is the GLS of  $\beta$ .

Also,  $K'\beta^\circ$  is the BLUE of  $K'\beta$  for  $K'$  being a set of estimable functions.

**Proof:**

The BLUE of  $K'\beta^\circ = K'(X'V^{-1}X)^{-1}X'V^{-1}y$ ,

but

$$V^{-1} = R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}$$

$\Rightarrow K'\beta^\circ$  from the MME is the BLUE of  $K'\beta$ .

**In addition, under normality,  $\beta^\circ$  is the MLE of  $\beta$ .** The proof is similar to the one for  $\beta^\circ = \text{GLS}$  of  $\beta$ .

(a.2)  $\hat{g}$  from the MME is the BLUP of  $g$

From the second equation of the MME we get:

$$(Z'R^{-1}Z + G^{-1})\hat{g} = Z'R^{-1}y - Z'R^{-1}X\beta^\circ.$$

$$\Rightarrow \hat{g} = (Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}(y - X\beta^\circ).$$

But the BLUP of  $g$  is:

$$\hat{g} = GZ'(ZGZ' + R)^{-1}(y - X\beta^\circ).$$

$\Rightarrow$  If  $\hat{g}$  from the MME is the BLUP of  $g$ , then:

$$\begin{aligned} GZ'(ZGZ' + R)^{-1} &= (Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1} \\ &= (Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}(ZGZ' + R)(ZGZ' + R)^{-1} \\ &= (Z'R^{-1}Z + G^{-1})^{-1}(Z'R^{-1}ZGZ' + Z')(ZGZ' + R)^{-1} \\ &= (Z'R^{-1}Z + G^{-1})^{-1}(Z'R^{-1}ZGZ' + G^{-1}GZ')(ZGZ' + R)^{-1} \\ &= (Z'R^{-1}Z + G^{-1})^{-1}(Z'R^{-1}Z + G^{-1})GZ'(ZGZ' + R)^{-1} \\ &= GZ'(ZGZ' + R)^{-1}. \end{aligned}$$

$\Rightarrow \hat{g}$  from the MME is the BLUP of  $g$ .

$$(b) \begin{bmatrix} K' & L' \end{bmatrix} \begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z + G^{-1} \end{bmatrix}^{-1} \begin{bmatrix} K \\ L \end{bmatrix}$$

is the EVP of the BLUP of  $w = K'\beta + L'g$ , i.e., the EVP of  $\hat{w} = K'\beta^\circ + L'\hat{g}$ .

Let a generalized inverse of the left hand side (LHS) of the MME be:

[11-8]

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} = \begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z + G^{-1} \end{bmatrix}.$$

By the rules of the inverse of a partitioned matrix the  $\{B_{ij}\}$  are:

$$\begin{aligned} B_{11} &= (X'R^{-1}X - X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}X)^{-1} \\ B_{12} &= -B_{11}X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1} \\ B_{22} &= (Z'R^{-1}Z + G^{-1})^{-1} + (Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}XB_{11}X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1} \end{aligned}$$

But

$$\begin{aligned} \text{var}(\hat{w} - w) &= \text{var}(K'\beta^\circ) + \text{var}(L'\hat{g} - L'g) \\ &\quad - \text{cov}(K'\beta^\circ, g'L) - \text{cov}(L'g, \beta^\circ K') \\ \text{var}(\hat{w} - w) &= K'(X'V^{-1}X)^{-1}K + L'[G - C'PC]L \\ &\quad - K'(X'V^{-1}X)^{-1}X'V^{-1}CL - L'C'V^{-1}X(X'V^{-1}X)^{-1}K. \end{aligned}$$

Thus, if the EVP of  $\hat{w}$  using  $\beta^\circ$  and  $\hat{g}$  from the MME is the EVP of the BLUP of  $w$ , then,

$$\begin{aligned} \text{(i)} \quad \text{var}(K'\beta^\circ) &= K'(X'V^{-1}X)^{-1}K &= K'B_{11}K \\ \text{(ii)} \quad \text{var}(L'\hat{g} - Lg) &= L'[G - C'PC]L &= L'B_{22}L \\ \text{(iii)} \quad -\text{cov}(K'\beta^\circ, g'L) &= -K'(X'V^{-1}X)^{-1}X'V^{-1}CL &= K'B_{12}L \end{aligned}$$

Thus,

$$\begin{aligned} \text{(i)} \quad K'(X'V^{-1}X)^{-1}K &= K'B_{11}K \\ &= K'[X'(R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1})X]^{-1}K \\ &= K'(X'V^{-1}X)^{-1}K \\ \text{(ii)} \quad L'[G - C'PC]L &= L'B_{22}L \\ &= L'[(Z'R^{-1}Z + G^{-1})^{-1}]L \end{aligned}$$

[11-9]

$$+ L'[(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1}XB_{11}X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}]L$$

By the formula of the inverse of a sum of matrices of the form  $(AVA' + B)^{-1}$ , the first term of  $B_{22}$  is equal to:

$$(Z'R^{-1}Z + G^{-1})^{-1} = G - GZ'(ZGZ' + R)^{-1}ZG = G - GZ'V^{-1}ZG$$

Also,

$$(Z'R^{-1}Z + G^{-1})^{-1}Z'R^{-1} = GZ'(ZGZ' + R)^{-1} = GZ'V^{-1}$$

as it was shown in (a.2). Thus,

$$\begin{aligned} L'B_{22}L &= L'[G - GZ'V^{-1}ZG + GZ'V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}ZG]L \\ &= L'[G - GZ'(V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1})ZG]L \\ &= L'[G - C'PC]L \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad -K'(X'V^{-1}X)^{-1}X'V^{-1}CL &= K'B_{12}L \\ &= K'[B_{11}X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}]L \\ &= K'[-(X'V^{-1}X)^{-1}X'V^{-1}ZG]L \\ &= -K'(X'V^{-1}X)^{-1}X'V^{-1}CL \end{aligned}$$

$$\Rightarrow \text{var}(\hat{w} - w) = [K' \quad L'] \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \begin{bmatrix} K \\ L \end{bmatrix} \text{ from the MME is the}$$

EVP of the BLUP of  $w$ .

### Derivation of the MME by maximizing the posterior density of $\theta|y$

Consider the joint density function:

$$f(y, \theta)$$

where

[11-10]

$y$  = observable random vector, and

$\theta$  = unobservable random vector.

### Definition

$\hat{\theta}$  is a maximum a posteriori (MAP) of  $\theta$  if it maximizes  $f(\theta|y)$ , the posterior density of  $\theta|y$ .

Thus, we need to maximize

$$f(\theta|y) = \frac{f(y|\theta)}{f(y)}$$

$$f(\theta|y) = \frac{f(y|\theta)f(\theta)}{f(y)}$$

with respect to  $\theta$ .

To maximize  $f(\theta|y)$  as such may be difficult. However, because  $\log(f(\theta|y))$  is an increasing function of  $f(\theta|y)$ , its maximum is the same as the maximum of  $f(\theta|y)$ .

Thus,

$$\log f(\theta|y) = \log f(y|\theta) + \log f(\theta) - \log f(y).$$

But **log f(y) is independent of θ**, so we only need to maximize

$$\Psi = \log f(y|\theta) + \log f(\theta),$$

which is the way C. R. Henderson derived the MME.

### Derivation of the MME by maximizing $\Psi$

Consider the mixed linear model:

$$y = X\beta + Zg + e$$

[11-11]

$$\begin{bmatrix} y \\ \beta \\ g \\ e \end{bmatrix} \sim MVN \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} XI\sigma_\beta^2 X' + ZGZ' + R & | & XI\sigma_\beta^2 & ZG & R \\ \hline & | & - & - & - \\ \sigma_\beta^2 IX' & | & I\sigma_\beta^2 & 0 & 0 \\ GZ' & | & 0 & G & 0 \\ R & | & 0 & 0 & R \end{bmatrix} \right\}.$$

The **conditional function of y given  $\theta$** , where  $\theta = \begin{bmatrix} \beta \\ g \end{bmatrix}$ , is:

$$f(y|\beta, g) \sim MVN \{ E[y|\beta, g], \text{var}(y|\beta, g) \}$$

where

$$E[y | \beta, g] = \begin{Bmatrix} 0 + [XI\sigma_\beta^2 & ZG] \begin{bmatrix} I\sigma_\beta^{-2} & 0 \\ 0 & G^{-1} \end{bmatrix} \begin{bmatrix} \beta - 0 \\ g - 0 \end{bmatrix} \end{Bmatrix}$$

$$E[y | \beta, g] = X\beta + Zg$$

and

$$\text{var}(y | \beta, g) = \left\{ (XI\sigma_\beta^2 X' + ZGZ' + R - [XI\sigma_\beta^2 & ZG] \begin{bmatrix} I\sigma_\beta^{-2} & 0 \\ 0 & G^{-1} \end{bmatrix} \begin{bmatrix} \sigma_\beta^2 IX' \\ GZ' \end{bmatrix}) \right\}$$

$$\text{var}(y | \beta, g) = R$$

$$\Rightarrow f(y|\beta, g) \sim MVN \{X\beta + Zg, R\}$$

The **joint function of  $\beta$  and  $g$**  is:

$$f(\beta, g) \sim MVN \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} I\sigma_\beta^2 & 0 \\ 0 & G \end{bmatrix} \right\}$$

Thus,

[11-12]

$$\Psi = \text{constants} - \frac{1}{2} \left\{ (\mathbf{y} - \mathbf{X}\beta - \mathbf{Z}\mathbf{g})' \mathbf{R}^{-1} (\mathbf{y} - \mathbf{X}\beta - \mathbf{Z}\mathbf{g}) + [\beta' \mathbf{g}]' \begin{bmatrix} \mathbf{I}_{\sigma_\beta^2} & 0 \\ 0 & \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \mathbf{g} \end{bmatrix} \right\}$$

$$\begin{aligned} \Psi = \text{constants} - \frac{1}{2} \{ & \mathbf{y}' \mathbf{R}^{-1} \mathbf{y} - 2 \mathbf{y}' \mathbf{R}^{-1} \mathbf{X} \beta - 2 \mathbf{y}' \mathbf{R}^{-1} \mathbf{Z} \mathbf{g} + \beta' \mathbf{X}' \mathbf{R}^{-1} \mathbf{X} \beta - 2 \beta' \mathbf{X}' \mathbf{R}^{-1} \mathbf{Z} \mathbf{g} \\ & + \mathbf{g}' \mathbf{Z}' \mathbf{R}^{-1} \mathbf{Z} \mathbf{g} + \beta' \mathbf{I} \sigma_\beta^2 \beta + \mathbf{g}' \mathbf{G}^{-1} \mathbf{g} \} \end{aligned}$$

$$\frac{\partial \Psi}{\partial \beta} = -\mathbf{X}' \mathbf{R}^{-1} \mathbf{y} + \mathbf{X}' \mathbf{R}^{-1} \mathbf{X} \beta + \mathbf{X}' \mathbf{R}^{-1} \mathbf{Z} \mathbf{g} + \mathbf{I} \sigma_\beta^2 \beta = 0$$

$$\frac{\partial \Psi}{\partial \mathbf{g}} = -\mathbf{Z}' \mathbf{R}^{-1} \mathbf{y} + \mathbf{Z}' \mathbf{R}^{-1} \mathbf{X} \beta + \mathbf{Z}' \mathbf{R}^{-1} \mathbf{Z} \mathbf{g} + \mathbf{G}^{-1} \mathbf{g} = 0$$

The resulting **pseudo-MME** are:

$$\begin{bmatrix} \mathbf{X}' \mathbf{R}^{-1} \mathbf{X} + \mathbf{I} \sigma_\beta^{-2} & \mathbf{X}' \mathbf{R}^{-1} \mathbf{Z} \\ \mathbf{Z}' \mathbf{R}^{-1} \mathbf{X} & \mathbf{Z}' \mathbf{R}^{-1} \mathbf{Z} + \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \mathbf{X}' \mathbf{R}^{-1} \mathbf{y} \\ \mathbf{Z}' \mathbf{R}^{-1} \mathbf{y} \end{bmatrix}$$

If we now **assume that**  $\sigma_\beta^2 \rightarrow \infty$  because we don't know whether the  $\beta$ 's are clustered around zero, i.e., we assume that we **know nothing about**  $\beta$ , then the pseudo-MME become the MME:

$$\begin{bmatrix} \mathbf{X}' \mathbf{R}^{-1} \mathbf{X} & \mathbf{X}' \mathbf{R}^{-1} \mathbf{Z} \\ \mathbf{Z}' \mathbf{R}^{-1} \mathbf{X} & \mathbf{Z}' \mathbf{R}^{-1} \mathbf{Z} + \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \mathbf{g} \end{bmatrix} = \begin{bmatrix} \mathbf{X}' \mathbf{R}^{-1} \mathbf{y} \\ \mathbf{Z}' \mathbf{R}^{-1} \mathbf{y} \end{bmatrix}.$$

Note that in Bayesian terms, if we use  $\sigma_\beta^2 \rightarrow \infty$ , we are using a so-called flat prior or improper prior.

### Remarks:

[1] With normality, the MAP estimator of  $\mathbf{w} = \mathbf{K}'\beta + \mathbf{L}'\mathbf{g}$ , with a flat prior for  $\sigma_\beta^2$  is the MLE of  $\mathbf{w}$ .

### Proof:

We want to predict  $\mathbf{w} = \mathbf{K}'\beta + \mathbf{L}'\mathbf{g}$

$$E[\mathbf{w} | \mathbf{y}] = \mathbf{K}'\beta + \mathbf{L}'\mathbf{C}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\beta)$$

[11-13]

$$= BX\beta + L'C'V^{-1}(y - X\beta) \quad \text{by property of BLUP [12.1].}$$

But **under normality** the MAP with a flat prior for  $\sigma_\beta^2$  yields the MME which yield the MLE of  $X\beta$ .

Thus,

$$\begin{aligned} E[w|y] &= BX\beta^\circ + L'C'V^{-1}(y - X\beta^\circ) \\ &= K'\beta^\circ + L'C'V^{-1}(y - X\beta^\circ) \end{aligned}$$

where  $\beta^\circ$  is the MAP estimator (and MLE) of  $\beta$ .

[2] The MAP estimator of  $w$  is also a Bayesian estimator of  $w$  with loss function:

$$m(\theta) = \begin{cases} 0 & \text{if } (\hat{\theta} - \theta) \leq \delta \\ 1 & \text{if } (\hat{\theta} - \theta) > \delta \end{cases}$$

### Properties of BLUP using the results from the MME

The solutions for  $\beta$  and  $g$  from the MME are:

$$\begin{bmatrix} \beta^\circ \\ \hat{g} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} \begin{bmatrix} X'R^{-1}y \\ Z'R^{-1}y \end{bmatrix}$$

$$\begin{bmatrix} \beta^\circ \\ \hat{g} \end{bmatrix} = \begin{bmatrix} B_{11}X'R^{-1} + B_{12}Z'R^{-1} \\ B_{12}'X'R^{-1} + B_{22}Z'R^{-1} \end{bmatrix} y$$

$$\begin{bmatrix} \beta^\circ \\ \hat{g} \end{bmatrix} = \begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} y$$

Thus,

$$\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} \begin{bmatrix} X & Z \end{bmatrix} = \begin{bmatrix} Q_1'X & Q_1'Z \\ Q_2'X & Q_2'Z \end{bmatrix}$$

[11-14]

$$\begin{aligned}
&= \begin{bmatrix} B_{11}X'R^{-1}X + B_{12}Z'R^{-1}X & | & B_{11}X'R^{-1}Z + B_{12}Z'R^{-1}Z \\ B_{12}'X'R^{-1}X + B_{22}Z'R^{-1}X & | & B_{12}'X'R^{-1}Z + B_{22}Z'R^{-1}Z \end{bmatrix} \\
&= \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} \begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z \end{bmatrix} \\
&= \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} \left\{ \begin{bmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z + G^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -G^{-1} \end{bmatrix} \right\} \\
&= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -B_{12}G^{-1} \\ 0 & -B_{22}G^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I & -B_{12}G^{-1} \\ 0 & I - B_{22}G^{-1} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[1] \text{ var}(K'\beta^\circ) &= \text{cov}(K'Q_1'y, y'Q_1K) \\
&= K'Q_1'(ZGZ' + R)Q_1K \\
&= K'[Q_1'ZGZ'Q_1 + B_{11}X'Q_1 + B_{12}Z'Q_1]K \\
&= K'[B_{12}G^{-1}GG^{-1}B_{12}' + B_{11} - B_{12}G^{-1}B_{12}']K \\
&= \text{K'B}_{11}\text{K} \\
&= K'[X'(R^{-1} - R^{-1}Z(Z'R^{-1}Z + G^{-1})Z'R^{-1})X]^-K \\
&= \text{K'}(X'V^{-1}X)^-K
\end{aligned}$$

$$\begin{aligned}
[2] \text{cov}(K'\beta^\circ, \hat{g}'L) &= \text{cov}(K'Q_1'y, y'Q_2L) \\
&= K'Q_1'(ZGZ' + R)Q_2L \\
&= K'[Q_1'ZGZ'Q_2 + B_{11}X'Q_2 + B_{12}Z'Q_2]L \\
&= K'[-B_{12}G^{-1}GI + B_{12}G^{-1}GG^{-1}B_{22} + 0 + B_{12}I - B_{12}G^{-1}B_{22}]L
\end{aligned}$$

[11-15]

$$= K'[0]L$$

$$= \mathbf{0}$$

$$[3] \text{ var}(X\beta^\circ) = \text{cov}(XQ_1'y, y'Q_1X')$$

$$= XB_{11}X'$$

$$= X(X'V^{-1}X)^{-1}X'$$

$$[4] \text{ cov}(X\beta^\circ, y') = \text{cov}(XQ_1'y, y')$$

$$= XQ_1'(ZGZ' + R)$$

$$= XQ_1'ZGZ' + XB_{11}X' + XB_{12}Z'$$

$$= -XB_{12}G^{-1}GZ' + XB_{11}X' + XB_{12}Z'$$

$$= XB_{11}X'$$

$$= X(X'V^{-1}X)^{-1}X'$$

$$[5] \text{ cov}(Z\hat{g}, y') = \text{cov}(ZQ_2'y, y')$$

$$= ZQ_2'(ZGZ' + R)$$

$$= ZQ_2'ZGZ' + ZB_{12}'X' + ZB_{22}Z'$$

$$= Z(I - B_{22}G^{-1})GZ' + ZB_{12}'X' + ZB_{22}Z'$$

$$= ZGZ' - ZB_{22}Z' + ZB_{12}'X' + ZB_{22}Z'$$

$$= ZGZ' + ZB_{12}'X'$$

$$= ZGZ' + ZC'V^{-1}X(X'V^{-1}X)^{-1}X'$$

$$[6] \text{ var}(L'\hat{g}) = \text{cov}(L'Q_2'y, y'Q_2L)$$

$$= L'[Q_2'(ZGZ' + R)Q_2]L$$

$$= L'[(I - B_{22}G^{-1})G(I - G^{-1}B_{22}) + Q_2'XB_{12} + Q_2'ZB_{22}]L$$

$$= L'[G - B_{22} - B_{22} + B_{22}G^{-1}B_{22} + 0 + B_{22} - B_{22}G^{-1}B_{22}]L$$

[11-16]

$$\begin{aligned}
 &= L'[G - B_{22}]L \\
 &= L'[G - (G - GZ'(V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1})ZG)]L \\
 &= L'[GZ'(V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1})ZG]L \\
 &= L'C'PCL
 \end{aligned}$$

$$\begin{aligned}
 [7] \text{ var}(\hat{w}) &= \text{var}(K'\beta^\circ + L'\hat{g}) \\
 &= \text{var}(K'\beta^\circ) + \text{var}(L'\hat{g}) \quad \text{by [2]} \\
 &= K'B_{11}K + L'[G - B_{22}]L.
 \end{aligned}$$

$$\begin{aligned}
 [8] \text{ cov}(L'\hat{g}, g'L) &= \text{cov}(L'Q_2'y, g'L) \\
 &= L'Q_2'ZGL \\
 &= L'[G - B_{22}G^{-1}G]L \\
 &= L'[G - B_{22}]L \\
 &= L'C'PCL \\
 &= \text{var}(L'\hat{g})
 \end{aligned}$$

$$\begin{aligned}
 [9] \text{ var}(L'\hat{g} - g'L) &= \text{var}(L'\hat{g}) + \text{var}(L'g) - 2\text{cov}(L'\hat{g}, g'L) \\
 &= \text{var}(L'g) - \text{var}(L'\hat{g}) \\
 &= L'GL - L'[G - B_{22}]L \\
 &= L'B_{22}L \\
 &= L'[G - C'PC]L
 \end{aligned}$$

$$\begin{aligned}
 [10] \text{ cov}(K'\beta^\circ, g'L) &= \text{cov}(K'Q_1'y, g'L) \\
 &= K'Q_1'ZGL \\
 &= -K'B_{12}G^{-1}GL
 \end{aligned}$$

[11-17]

$$\begin{aligned}
 &= -K'B_{12}L \\
 &= K'[-B_{11}X'R^{-1}Z(Z'R^{-1}Z + G^{-1})^{-1}]L \\
 &= -K'[(X'V^{-1}X)^{-1}X'V^{-1}ZG]L \\
 &= -K'(XV^{-1}X)^{-1}X'V^{-1}CL
 \end{aligned}$$

$$\begin{aligned}
 [11] \quad \text{var}(\hat{w} - w) &= \text{var}(K'\beta^\circ + L'\hat{g} - K'\beta - L'g) \\
 &= \text{var}(K'\beta^\circ) + \text{var}(L'\hat{g} - L'g) - \text{cov}(K'\beta^\circ, g'L) - \text{cov}(L'g, \beta^\circ'K) \\
 &= K'B_{11}K + L'B_{22}L - K'B_{12}L - L'B_{12}'K \\
 &= K'[(X'V^{-1}X)^{-1}K + L'[G - C'PC]L - K'(X'V^{-1}X)^{-1}X'V^{-1}CL \\
 &\quad - L'C'V^{-1}X(X'V^{-1}X)^{-1}K]
 \end{aligned}$$

$$\begin{aligned}
 [12] \quad \text{var}(\hat{e}) &= \text{var}(y - X\beta^\circ - Z\hat{g}) \\
 &= \text{var}(y) + \text{var}(X\beta^\circ) + \text{var}(Z\hat{g}) - \text{cov}(y, \beta^\circ'X) - \text{cov}(X\beta^\circ, y') \\
 &\quad - \text{cov}(y, \hat{g}'Z) - \text{cov}(Z\hat{g}, y') - \text{cov}(X\beta^\circ, \hat{g}'Z) - \text{cov}(Z\hat{g}, \beta^\circ'X) \\
 &= (ZGZ' + R) + XB_{11}X' + Z[G - B_{22}]Z' - XB_{11}X' - XB_{11}X' \\
 &\quad - ZGZ' - ZGZ' - XB_{12}Z' - ZB_{12}'X' \\
 &= R - XB_{11}X' - XB_{12}Z' - ZB_{12}'X' - ZB_{22}Z' \\
 &= R - X'[(X'V^{-1}X)^{-1}X - X(X'V^{-1}X)^{-1}X'V^{-1}CZ' \\
 &\quad - ZC'V^{-1}X(X'V^{-1}X)^{-1}X' - Z[G - C'PC]Z']
 \end{aligned}$$

$$\begin{aligned}
 [13] \quad \text{var}(\hat{e} - e) &= \text{var}(\hat{e}) + \text{var}(e) - 2\text{cov}(\hat{e}, e) \\
 &= \text{var}(e) - \text{var}(\hat{e}) \\
 &= R - [R - XB_{11}X' - XB_{12}Z' - ZB_{12}'X' - ZB_{22}Z'] \\
 &= XB_{11}X' + XB_{12}Z' + ZB_{12}'X' + ZB_{22}Z' \\
 &= X(X'V^{-1}X)^{-1}X' + X(X'V^{-1}X)^{-1}X'V^{-1}CZ'
 \end{aligned}$$

[11-18]

$$+ ZC'V^{-1}X(X'V^{-1}X)^{-1}X' + Z[G - C'PC]Z'$$

### Example of the BLUP of $w = K'\beta + L'g$ using the general form and the MME

A sire ( $s_0$ ) has 3 sons ( $s_1, s_2, s_3$ ) with progeny in 2 herds. The following weaning weights (kg) were measured in these progeny:

Herd	Sires			Herd Total
	1	2	3	
1	248 256	296 282	265 274	
Sire $\times$ Herd	504	578	539	1621
2	260 252	300 285	295 290	
Sire $\times$ Herd	512	585	585	1682
Sire Total	1016	1163	1124	3303

Let

$$\text{phenotypic variance} = 108.60 \text{ kg}^2 = \sigma_p^2$$

$$\text{additive genetic variance} = 21.72 \text{ kg}^2 = \sigma_A^2$$

$$\text{residual variance} = 86.88 \text{ kg}^2 = \sigma_e^2$$

$$\Rightarrow h^2 = \frac{\sigma_A^2}{\sigma_p^2} = 0.20$$

Let the equation for a record be:

$$y_{ijk} = \text{herd}_i + \text{sire}_j + \text{residual}_{ijk}$$

Thus, the linear model, in matrix notation, is:

$$y = Xh + Zs + e$$

[11-19]

$$\begin{bmatrix} y \\ s \\ e \end{bmatrix} \sim \left\{ \begin{bmatrix} Xh \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ZGZ + I\sigma_e^2 & | & ZG & I\sigma_e^2 \\ ----- & | & -- & -- \\ GZ' & | & G & 0 \\ I\sigma_e^2 & | & 0 & I\sigma_e^2 \end{bmatrix} \right\}$$

where

$$G = A (\frac{1}{4} \sigma_A^2)$$

$$G = A (\sigma_s^2)$$

$$G = \begin{bmatrix} 1.00 & 0.25 & 0.25 \\ 0.25 & 1.00 & 0.25 \\ 0.25 & 0.25 & 1.00 \end{bmatrix} (5.4300) = \begin{bmatrix} 5.4300 & 1.3575 & 1.3575 \\ 1.3575 & 5.4300 & 1.3575 \\ 1.3575 & 1.3575 & 5.4300 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} y \\ 248 \\ 256 \\ 296 \\ 282 \\ 265 \\ 274 \\ 260 \\ 252 \\ 300 \\ 285 \\ 295 \\ 290 \end{bmatrix} = \begin{bmatrix} X \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} h + \begin{bmatrix} Z \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} s + \begin{bmatrix} e \\ e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} e \\ e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \end{bmatrix}$$

Let

[11-20]

$$K' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$L' = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

**(a) BLUP of w using the general form**

$$\hat{w} = K'h^\circ + L'\hat{s}$$

where

$$h^\circ = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$\hat{s} = C'V^{-1}(y - Xh^\circ)$$

Thus, we need to compute:

$$(a.1) \quad V^{-1} = (ZGZ' + I\sigma_e^2)^{-1}$$

$$V^{-1} = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 5.4300 & 1.3575 & 1.3575 \\ 1 & 0 & 0 & 5.4300 & 1.3575 & 1.3575 \\ 0 & 1 & 0 & Sym & & 5.4300 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 1 & & & \\ \hline --- & --- & --- & Z' & + & I(86.88) \\ 1 & 0 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 1 & & & \end{array} \right]^{-1}$$

$$(a.2) \quad C' = GZ'$$

[11-21]

$$C' = \begin{bmatrix} 5.4300 & 1.3575 & 1.3575 \\ & 5.4300 & 1.3575 \\ & & 5.4300 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & | & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & | & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & | & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$(a.3) \quad X'V^{-1}$$

$$(a.4) \quad (X'V^{-1}X)^{-1}$$

$$(a.5) \quad h^\circ = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$h^\circ = \begin{bmatrix} 270.1667 \\ 280.3333 \end{bmatrix}$$

$$(a.6) \quad C'V^{-1}$$

$$(a.7) \quad Xh^\circ$$

$$(a.8) \quad y - Xh^\circ$$

$$(a.9) \quad \hat{s} = C'V^{-1}(y - Xh^\circ)$$

$$\hat{s} = \begin{bmatrix} -3.3533 \\ 2.4474 \\ 0.9079 \end{bmatrix}$$

$$(a.10) \quad \hat{w} = K'h^\circ + L'\hat{s}$$

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} h_1^\circ \\ h_2^\circ \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} -10.1666 \\ -10.1666 \\ -10.1666 \end{bmatrix} + \begin{bmatrix} -5.8027 \\ -4.2632 \\ 1.5395 \end{bmatrix}$$

[11-22]

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} -16.4693 \\ -14.9298 \\ -8.6271 \end{bmatrix}$$

$$\begin{aligned}
 (a.11) \quad \text{var}(L' \hat{s} - L's) &= L'[G - C'PC]L \\
 &= L'[G - C'(V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1})C]L \\
 &= \begin{bmatrix} 6.86 & 3.23 & -3.43 \\ & 6.86 & 3.43 \\ \text{Sym.} & & 6.86 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 (a.12) \quad \text{var}(K'\beta^\circ) &= K'(X'V^{-1}X)^{-1}K \\
 &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 17.195 & 2.715 \\ 2.715 & 17.195 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \\
 &= J_3 * (28.96)
 \end{aligned}$$

### (b) BLUP of w using the MME

$$\begin{bmatrix} X' I \sigma_e^{-2} X & | & X' I \sigma_e^{-2} Z \\ \hline \hline & | & \end{bmatrix} \begin{bmatrix} h^\circ \\ \hline \hline \hat{s} \end{bmatrix} = \begin{bmatrix} X' I \sigma_e^{-2} y \\ \hline \hline Z' I \sigma_e^{-2} y \end{bmatrix}$$

Multiplying the MME by  $\sigma_e^2$  yields:

$$\begin{bmatrix} X' X & | & X' Z \\ \hline \hline & | & \end{bmatrix} \begin{bmatrix} h^\circ \\ \hline \hline \hat{s} \end{bmatrix} = \begin{bmatrix} X' y \\ \hline \hline Z' y \end{bmatrix}$$

Here, we need to compute:

[11-23]

$$(b.1) \quad X'X = \begin{bmatrix} n_{1\bullet} & 0 \\ 0 & n_{2\bullet} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$(b.2) \quad X'Z = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$(b.3) \quad Z'Z = \begin{bmatrix} n_{\bullet 1} & 0 & 0 \\ 0 & n_{\bullet 2} & 0 \\ 0 & 0 & n_{\bullet 3} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$(b.4) \quad G^{-1}\sigma_e^2 = \begin{bmatrix} 5.4300 & 1.3575 & 1.3575 \\ & 5.4300 & 1.3575 \\ \text{Sym} & & 5.4300 \end{bmatrix}^{-1} \quad (86.88)$$

$$= \begin{bmatrix} 0.20462 & -0.04092 & -0.04092 \\ -0.04092 & 0.20462 & -0.04092 \\ -0.04092 & -0.04092 & 0.20462 \end{bmatrix} \quad (86.88)$$

$$(b.5) \quad X'y = \begin{bmatrix} y_{1\bullet\bullet} \\ y_{2\bullet\bullet} \end{bmatrix}$$

$$= \begin{bmatrix} 1621 \\ 1682 \end{bmatrix}$$

[11-24]

$$(b.6) \quad Z'y = \begin{bmatrix} y_{\bullet 1\bullet} \\ y_{\bullet 2\bullet} \\ y_{\bullet 3\bullet} \end{bmatrix} = \begin{bmatrix} 1016 \\ 1163 \\ 1124 \end{bmatrix}$$

The MME are:

$$\left[ \begin{array}{ccc|ccc} 6 & 0 & | & 2 & 2 & 2 \\ & 6 & | & 2 & 2 & 2 \\ \hline & & | & \dots & \dots & \dots \\ & & | & 4+17.78 & -3.56 & -3.56 \\ Sym & & | & & 4+17.78 & -3.56 \\ & & | & & & 4+17.78 \end{array} \right] \begin{bmatrix} h_1^o \\ h_2^o \\ \vdots \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} 1621 \\ 1682 \\ \vdots \\ 1016 \\ 1163 \\ 1124 \end{bmatrix}$$

The vector of solutions is:

$$\begin{bmatrix} h_1^o \\ h_2^o \\ \vdots \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \left[ \begin{array}{cc|ccccc} 0.19792 & 0.03125 & | & -0.3125 & -0.3125 & -0.3125 & 1621 \\ 0.19792 & & | & -0.3125 & -0.3125 & -0.3125 & 1682 \\ \hline & & | & \dots & \dots & \dots & \vdots \\ & & | & 0.05757 & 0.01809 & 0.01809 & 1016 \\ Sym & & | & & 0.05757 & 0.01809 & 1163 \\ & & | & & & 0.05757 & 1124 \end{array} \right]$$

$$\begin{bmatrix} h_1^o \\ h_2^o \\ \vdots \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} 270.17 \\ 280.33 \\ \vdots \\ -3.36 \\ 2.45 \\ 0.91 \end{bmatrix}$$

The predictor of w,  $\hat{w} = K'h^o + L'\hat{s}$ , is:

[11-25]

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} h_1^\circ \\ h_2^\circ \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} -10.1666 \\ -10.1666 \\ -10.1666 \end{bmatrix} + \begin{bmatrix} -5.8027 \\ -4.2632 \\ 1.5395 \end{bmatrix}$$

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \begin{bmatrix} -16.4693 \\ -14.9298 \\ -8.6271 \end{bmatrix}$$

The EVP of  $L' \hat{s}$  is:

$$\text{var}(L' \hat{s} - L's) = L'B_{22}L * \sigma_e^2$$

$$\begin{aligned} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0.05757 & 0.01809 & 0.01809 \\ 0.05757 & 0.01809 & 0.05757 \\ \text{Sym.} & 0.05757 & 0.05757 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} * 86.88 \\ &= \begin{bmatrix} 6.86 & 3.23 & -3.43 \\ 6.86 & 3.43 & \\ \text{Sym.} & 6.86 & \end{bmatrix} \end{aligned}$$

The variance of  $K'\beta^\circ$  is:

$$\text{var}(K'\beta^\circ) = K'B_{11}K * \sigma_e^2$$

$$\begin{aligned} &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.19792 & 0.03125 \\ 0.03125 & 0.19792 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} * 86.88 \\ &= J_3 * (28.96) \end{aligned}$$

The EVP of  $\hat{w}$  is:

[11-26]

$$\begin{aligned}\text{var}(\hat{w} - w) &= [K' \ L'] \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} K \\ L \end{bmatrix} * 86.88 \\ &= \begin{bmatrix} 35.82 & 32.39 & 25.53 \\ & 35.82 & 32.39 \\ & & 35.82 \end{bmatrix}\end{aligned}$$

The BLUP of  $e = y - Xh - Zs$  is:

$$\hat{e} = \begin{array}{c|c|c|c} 248 & 270.17 & -3.36 & -18.81 \\ 256 & 270.17 & -3.36 & -10.81 \\ 296 & 270.17 & 2.45 & 23.39 \\ 282 & 270.17 & 2.45 & 9.39 \\ 265 & 270.17 & 0.91 & -6.07 \\ 274 & 270.17 & 0.91 & 2.93 \\ \hline --- & --- & --- & --- \\ 260 & 280.33 & -3.36 & -16.98 \\ 252 & 280.33 & -3.36 & -24.98 \\ 300 & 280.33 & 2.45 & 17.22 \\ 285 & 280.33 & 2.45 & 2.22 \\ 295 & 280.33 & 0.91 & 13.76 \\ 290 & 280.33 & 0.91 & 8.76 \end{array}$$

The EVP of  $\hat{e}$  is:

$$\text{var}(\hat{e} - e) = [XB_{11}X' + XB_{12}Z' + ZB_{12}'X' + ZB_{22}Z'] * \sigma_e^2$$

The diagonal elements of the matrix  $\text{var}(\hat{e} - e)$  are all equal. This is because all calves have the same amount of information and their sires are paternal half-sibs.

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