ANIMAL BREEDING NOTES

CHAPTER 12

ESTIMATION, ESTIMABILITY AND SOLVING THE MIXED MODEL EQUATIONS WHEN THEY ARE NOT FULL-RANK

Methods of Estimation

Three methods of estimation are described here:

1) Generalized least squares,

2) Maximum likelihood, and

3) Best linear unbiased estimation.

Consider the general linear model:

$$y = X\beta + \varepsilon$$

$$\begin{bmatrix} y \\ \varepsilon \end{bmatrix} \sim \left\{ \begin{bmatrix} X\beta \\ 0 \end{bmatrix}, \begin{bmatrix} V & V \\ V & V \end{bmatrix} \right\}$$

where

y = $[y_1\ y_2\ \ldots\ y_n]'$ is a vector of observations,

β = $[\beta_1\ \beta_2\ \ldots\ \beta_I]'$ is a vector of unknown fixed effects,

ε = $[\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_n]'$ is a vector of residual effects, i.e., $\varepsilon = (y - X\beta)$,

X = $[X_1\ X_2\ \ldots\ X_I)'$ is a known incidence matrix relating elements of y to elements of β.

Generalized Least Squares (GLS)

We want to estimate β based on y, using an estimator that minimizes the generalized sum of squares of deviations of the observations from their expected values. Let $\beta°$ be such an estimator. Thus, we want $\beta°$ to minimize

$$\varepsilon' V^{-1} \varepsilon \quad = \quad (y - X\beta)' V^{-1} (y - X\beta)$$

with respect to β. The resulting GLS estimator of β is:

$$\beta° \quad = \quad (X' V^{-1} X)^{-} X' V^{-1} y$$

Proof:

$$L \quad \equiv \quad (y - X\beta)' V^{-1} (y - X\beta)$$

$$L \quad = \quad y' V^{-1} y - y' V^{-1} X\beta - \beta' X' V^{-1} y + \beta' X' V^{-1} X\beta$$

$$\frac{\partial L}{\partial \beta} = \quad -2X' V^{-1} y + 2XV^{-1} X\beta = \quad 0$$

$$\Rightarrow \qquad \beta° \quad = \quad (X' V^{-1} X)^{-} X' V^{-1} y$$

Note that **if $V^{-1} = I\sigma^2$**, the **GLS estimator of β becomes**:

$$\beta° \quad = \quad (X'X)^{-} X' y$$

which is called the **ordinary least squares (OLS) estimator of β**. The OLS can also be obtained by minimizing

$$(y - X\beta)'(y - X\beta)$$

with respect to β.

Maximum Likelihood (ML)

Assuming normality:

$$\begin{bmatrix} y \\ \varepsilon \end{bmatrix} \sim MVN \left\{ \begin{bmatrix} X\beta \\ 0 \end{bmatrix}, \begin{bmatrix} V & V \\ V & V \end{bmatrix} \right\}$$

Thus, the ε's are assumed to be normally distributed with mean zero and covariance matrix V. The β will be estimated by maximizing the likelihood function of the sample of observations represented in vector y. The **likelihood function of y** (i.e., the **function of parameters β and V given a data vector y**; Searle et al., 1992, pg. 79) is:

$$L \quad = \quad L(\beta, V \mid y) \quad = \quad (2\pi)^{-\frac{n}{2}} \left|V\right|^{-½} \exp\{-½(y - X\beta)'V^{-1}(y - X\beta)\}$$

Maximizing L with respect to β yields the ML estimator (MLE) of β, i.e., β°, where

$$\beta° \quad = \quad (X'V^{-1}X)^{-}X'V^{-1}y$$

Proof:

Maximizing Ψ = log L is equivalent to maximizing L because both functions are continuous and increasing and have the same maximum given by β. Thus,

$$\Psi \quad = \quad -\frac{n}{2}\log 2\pi - ½\log\left|V\right| - ½(y'V^{-1}y - y'V^{-1}X\beta - \beta'X'V^{-1}y + \beta'X'V^{-1}X\beta)$$

$$\frac{\partial\Psi}{\partial\beta} \quad = \quad -X'V^{-1}y + X'V^{-1}X\beta \quad = \quad 0$$

$$\Rightarrow \qquad \beta° \quad = \quad (X'V^{-1}X)^{-}X'V^{-1}y$$

Best Linear Unbiased Estimation (BLUE)

Given

$$\begin{bmatrix} y \\ \varepsilon \end{bmatrix} \sim \left\{ \begin{bmatrix} X\beta \\ 0 \end{bmatrix}, \begin{bmatrix} V & V \\ V & V \end{bmatrix} \right\}$$

and K′β estimable, we want to estimate K′β using an estimator (K′β°) that:

(i) It is linear in y, i.e.,

$$K'\beta° \quad = \quad a + By$$

(ii) It has minimum mean square error of estimation, i.e., K′β° is chosen to minimize:

$$E[(K'\beta^\circ - K'\beta)'(K'\beta^\circ - K'\beta)] \quad = \quad E[(a + By - K'\beta)'(a + By - K'\beta)]$$

(iii) It is unbiased, i.e.,

$$E[K\beta^\circ] = E[a + By] = K'\beta$$

$$= a + BX\beta = K'\beta$$

$$\Rightarrow \quad a = 0 \quad \text{and} \quad BX = K' \quad \Leftrightarrow \quad (BX - K') = 0$$

The resulting BLUE of $K'\beta$ is:

$$K'\beta^\circ = K'(X'V^{-1}X)^{-}X'V^{-1}y$$

Proof:

Minimize L with respect to a, B and M', where

$$L = E[(a + By - K'\beta)'(a + By - K'\beta) + tr(2M'(BX - K'))$$

where $2M'$ is a matrix of Lagrange multipliers.

$$L = E[a'a + a'By - a'K'\beta + y'B'a + y'B'By - y'B'K'\beta - \beta'Ka - \beta'KBy$$
$$+ \beta'KK'\beta + tr(2M'(BX - K'))]$$

$$L = a'a + 2a'BX\beta - 2a'K'\beta + trB'BV + \beta'X'B'BX\beta - 2\beta'X'B'K'\beta - \beta'KK'\beta$$
$$+ tr(2M'BX - 2M'K')$$

$$\frac{\partial L}{\partial a} = 2a + 2BX\beta - 2K'\beta = 0$$

$$\Rightarrow \quad a + BX\beta = K'\beta$$

$$\frac{\partial L}{\partial M'} = 2X'B' - 2K = 0$$

$$\Rightarrow \quad X'B' = K$$

$$\Rightarrow \quad BX = K'$$

Substituting $BX = K'$ in the equation for a yields:

$$a + BX\beta = K'\beta$$

$$\Rightarrow \quad a = K'\beta - BX\beta$$

$$\Rightarrow \quad a = BX\beta - BX\beta$$

$$\Rightarrow \quad a = 0$$

$$\frac{\partial L}{\partial B} = 2a\beta'X' + 2BV + 2BXB\beta'X' - 2K'\beta\beta'X' - 2MX' = 0$$

But

$$a = 0 \text{ and } BX = K',$$

Thus,

$$0 + BV + K'\beta\beta'X' - K'\beta\beta'X' + MX' = 0$$

$\Rightarrow$ We need to solve the following set of equations for M and B:

$$BV + MX' = 0 \qquad (1)$$

$$BX = K' \qquad (2)$$

From (1),

$$B = -MX'V^{-1} \qquad (3)$$

From (3) and (2),

$$-MX'V^{-1}X = K'$$

$$\Rightarrow \quad M = -K'(X'V^{-1}X)^{-} \qquad (4)$$

From (4) and (3),

$$B = K'(X'V^{-1}X)^{-}X'V^{-1} \qquad (5)$$

$\Rightarrow$ the BLUE of $K'\beta$ is:

$$K'\beta^{\circ} = a + By$$

$$K'\beta^{\circ} = 0 + K'(X'V^{-1}X)^{-}X'V^{-1}y$$

$$K'\beta^\circ = K'(X'V^{-1}X)^{-}X'V^{-1}y$$

Properties of K′β°, the BLUE of K′β.

[1] $$E[K'\beta^\circ] = E[K'(X'V^{-1}X)^{-}X'V^{-1}y]$$
$$= K'(X'V^{-1}X)^{-}X'V^{-1}X\beta$$
$$= K'\beta$$

$\Rightarrow$ $K'\beta^\circ$ is unbiased.

Note: unbiasedness was **required** for the BLUE of $K'\beta$, but **not** for the GLS of $K'\beta$.

[2] $$\text{var}(K'\beta^\circ) = K'(X'V^{-1}X)^{-}X'V^{-1}\,\text{cov}(y, y')V^{-1}X(X'V^{-1}X)^{-}K$$
$$= K'(X'V^{-1}X)^{-}K$$

[3] $$\text{var}(X\beta^\circ) = X(X'V^{-1}X)^{-}X'$$

[4] $$\text{cov}(X\beta^\circ, y') = X(X'V^{-1}X)^{-}X'V^{-1}\text{cov}(y, y')$$
$$= X(X'V^{-1}X)^{-}X'$$
$$= \text{var}(X\beta^\circ)$$

[5] $$\text{cov}(X\beta^\circ, \varepsilon) = X(X'V^{-1}X)^{-}X'V^{-1}\text{cov}(y, y' - \beta'X')$$
$$= X(X'V^{-1}X)^{-}X'$$
$$= \text{var}(X\beta^\circ)$$
$$= \text{cov}(X\beta^\circ, y')$$

[6] $$\text{cov}(X\beta^\circ, \hat{\varepsilon}) = \text{cov}(X\beta^\circ, y' - \beta^{\circ\prime}X')$$
$$= \text{cov}(X\beta^\circ, y') - \text{cov}(X\beta^\circ, \beta^{\circ\prime}X')$$
$$= \text{var}(X\beta^\circ) - \text{var}(X\beta^\circ)$$
$$= 0$$

Estimability and estimable functions

The general linear model is:

$$y = X\beta + \varepsilon$$

$$y \sim (X\beta, V)$$

Estimability: an element of β or a linear combination of the elements of β is estimable if it can be estimated unbiasedly.

Estimable function: a linear function of the elements of β is estimable if it is identical to a linear function of the expected values of the vector of observations y.

Thus,

$$\begin{aligned} K'\beta \text{ is estimable} &\Leftrightarrow T'E[y] \\ &\Leftrightarrow T'X\beta \\ &\Leftrightarrow K' = T'X \end{aligned}$$

Properties of estimable functions

[1] The expected value of an observation is estimable.

Proof:

By definition $K'\beta = T'X\beta$. Thus, for one record:

$$K'\beta = [1\ \ 0 \ldots 1\ \ 0 \ldots 1]\beta$$

$$T'X\beta = [1\ \ 0 \ldots 0]\begin{bmatrix} 1 & 0 & \ldots & 1 & 0 & \ldots 1 \\ & \vdots & & & \vdots & \vdots \end{bmatrix}\beta$$

$$T'X\beta = [1\ \ 0 \ldots 1\ \ 0 \ldots 1]\beta$$

$$T'X\beta = K'\beta$$

[2] Linear combinations of estimable functions are estimable.

Proof:

$K'\beta$ is estimable

$\Rightarrow \quad K'\beta = T'X\beta$

Consider $L'K'\beta$,

$$L'(K'\beta) = L'(T'X\beta)$$

$$K^*\beta = T^*X\beta$$

$\Rightarrow \quad L'K'\beta$ is estimable.

[3] The general form of a set of estimable functions is $K' = T'X$.

Proof:

By definition, $K'\beta = T'X\beta$,

$\Rightarrow \quad K'\beta$ is estimable $\Leftrightarrow K' = T'X$

$\Rightarrow$ the general form of $K' = T'X$ indicates that a linear function of the β's will be estimable if and only if the corresponding row of K' is in the row space of X.

[4] If $K'\beta$ is estimable, then $K'\beta^\circ$ is invariant to the value of β°.

Proof:

$$K'\beta^\circ = K'(X'V^{-1}X)^{-}X'V^{-1}y$$

$$K'\beta^\circ = T'X(X'V^{-1}X)^{-}X'V^{-1}y$$

But $(X'V^{-1}X)^{-}X'V^{-1}y$ is invariant to $(X'V^{-1}X)^{-}$

$\Rightarrow \quad K'\beta^\circ$ is invariant to β°

Also, note that

$$E[K'\beta^\circ] = E[T'X\beta^\circ]$$

$$= T'X(X'V^{-1}X)^{-}X'V^{-1}X\beta$$

$= \quad T'X\beta$ regardless of the value of $\beta°$ when $K'\beta$ is estimable.

[5] The BLUE of the estimable function $K'\beta$ is $K'\beta°$.

Proof: See derivation of BLUE.

Checking for estimability

[1] Given a set of linear functions of the elements of β, i.e., $K'\beta$, look for a set of linear combinations of the expected values of records $T'E[y]$. If $K'\beta$ is estimable, then

$$K'\beta = T'E[y] \text{ for some } T',$$

i.e.,

$$k_i'\beta \text{ is estimable } \Leftrightarrow k_i'\beta = t_i'E[y],$$

by properties [1] and [2] of estimable functions; otherwise $k_i'\beta$ is not estimable.

[2] If $(X'V^{-1}X)^-$ and $(X'V^{-1}X)$ are available, then $K'\beta$ is estimable if

$$K'(X'V^{-1}X)^-(X'V^{-1}X) = K'$$

by properties [3] and [4] of estimable functions.

Note that if $K' = I$,

$$K'(X'V^{-1}X)^-(X'V^{-1}X) = E[\beta°]$$

which may be of interest to determine what linear combinations of the β's are contained in $\beta°$.

[2.1] If

$$(X'V^{-1}X)^- = \begin{bmatrix} (X_1'V^{-1}X_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

then,

$$E[\beta°] = \begin{bmatrix} (X_1'V^{-1}X_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1'V^{-1}X_1 & X_1'V^{-1}X_2 \\ X_2'V^{-1}X_1 & X_2'V^{-1}X_2 \end{bmatrix} \beta$$

$$E[\beta°] = \begin{bmatrix} I & (X_1'V^{-1}X_1)^{-1}(X_1'V^{-1}X_2) \\ 0 & 0 \end{bmatrix} \beta$$

But X_2 is a linear combination of the columns of X_1, i.e., $X_2 = X_1L$.

$$\Rightarrow \qquad E[\beta°] = \begin{bmatrix} I & (X_1'V^{-1}X_1)^{-}(X_1'V^{-1}X_1L) \\ 0 & 0 \end{bmatrix} \beta$$

$$E[\beta°] = \begin{bmatrix} I & L \\ 0 & 0 \end{bmatrix} \beta$$

[2.2] If

$$(X'V^{-1}X)^{-} = \begin{bmatrix} X_1'V^{-1}X_1 & X_1'V^{-1}X_1L \\ L'X_1'V^{-1}X_1 & L'X_1'V^{-1}X_1L \end{bmatrix}^{-} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix}$$

then,

$$E[\beta°] = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} \begin{bmatrix} X_1'V^{-1}X_1 & X_1'V^{-1}X_1L \\ L'X_1'V^{-1}X_1 & L'X_1'V^{-1}X_1L \end{bmatrix} \beta$$

$$E[\beta°] = \begin{bmatrix} (B_{11} + B_{12}L')X_1'V^{-1}X_1 & (B_{11} + B_{12}L')X_1'V^{-1}X_1L \\ (B_{12}' + B_{22}L')X_1'V^{-1}X_1 & (B_{12}' + B_{22}L')X_1'V^{-1}X_1L \end{bmatrix} \beta$$

Examples of estimable functions

[1] Models with a general mean and a single fixed effect.

$y_{ij} = \mu + a_i + \varepsilon_{ij}$

$y \sim (\mu + a_i, \sigma)$ and $\mathrm{cov}(\varepsilon_{ij}, \varepsilon_{i'j'}) \neq 0$

[1.1] $\mu + a_i$ is estimable.

Proof: $E[y_{ij}] = \mu + a_i$

[1.2] $a_i - a_{i'}$ is estimable.

Proof: $[1\ -1]\begin{bmatrix} E[y_{ij}] \\ E[y_{i'j'}] \end{bmatrix} = \mu + a_i - \mu - a_{i'} = a_i - a_{i'}$

[1.3] μ is **not** estimable and the a_i are **not** estimable.

Proof: there is not a t' such that $t'E[y_{ij}] = \mu$ or a_i .

[2] Models with a general mean and two fixed effects.

$y_{ijk} = \mu + a_i + b_j + \varepsilon_{ijk}$

$y \sim (\mu + a_i + b_j, \sigma)$ and $\text{cov}(\varepsilon_{ijk}, \varepsilon_{i'j'k'}) \neq 0$.

[2.1] $\mu + a_i + b_j$ is estimable.

Proof: $E[y_{ijk}] = \mu + a_i + b_j$

[2.2] $a_i - a_{i'}$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'jk'}] \end{bmatrix} = \mu + a_i + b_j - \mu - a_{i'} - b_j$$

$$= a_i - a_{i'}$$

[2.3] $b_j - b_{j'}$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{ij'k'}] \end{bmatrix} = \mu + a_i + b_j - \mu - a_i - b_{j'}$$

$$= \quad b_j - b_{j'}$$

[2.4] **Non-estimable** functions are:

(a) μ

(b) a_i

(c) b_j

(d) $\mu + a_i$

(e) $\mu + b_j$

[3] Models with a general mean, two fixed effects and an interaction between these two fixed effects.

$$y_{ijk} \quad = \quad \mu + a_i + b_j + ab_{ij} + \varepsilon_{ijk}$$

$$y \quad \sim \quad (\mu + a_i + b_j + ab_{ij}, \sigma) \;\text{ and } \operatorname{cov}(\varepsilon_{ijk}, \varepsilon_{i'j'k'}) \neq 0 \,.$$

[3.1] $\mu + a_i + b_j + ab_{ij}$ is estimable.

Proof:

$$E[y_{ijk}] = \mu + a_i + b_j + ab_{ij}$$

[3.2] $(a_i - a_{i'}) + (ab_{ij} - ab_{i'j})$ is estimable.

Proof:

$$[1 \; -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'jk'}] \end{bmatrix} \quad = \quad \mu + a_i + b_j + ab_{ij} - \mu - a_{i'} - b_j - ab_{i'j}$$

$$= \quad (a_i - a_{i'}) + (ab_{ij} - ab_{i'j})$$

[3.3] $(b_j - b_{j'}) + (ab_{ij} - ab_{ij'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{ij'k'}] \end{bmatrix} = \mu + a_i + b_j + ab_{ij} - \mu - a_i - b_{j'} - ab_{ij'}$$

$$= (b_j - b_{j'}) + (ab_{ij} - ab_{ij'})$$

[3.4] $(a_i - a_{i'}) + (b_j - b_{j'}) + (ab_{ij} - ab_{i'j'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'j'k'}] \end{bmatrix} = \mu + a_i + b_j + ab_{ij} - \mu - a_{i'} - b_{j'} - ab_{i'j'}$$

$$= (a_i - a_{i'}) + (b_j - b_{j'}) + (ab_{ij} - ab_{i'j'})$$

[3.5] $(a_i - a_{i'}) + \sum_{j=1}^{J} (ab_{ij} - ab_{i'j})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E\left[\sum_{j=1}^{J} y_{ijk}\right] \\ E\left[\sum_{j=1}^{J} y_{i'jk'}\right] \end{bmatrix} = \mu + a_i + \sum_{j=1}^{J} b_j + \sum_{j=1}^{J} ab_{ij} - \mu - a_{i'}$$

$$- \sum_{j=1}^{J} b_j - \sum_{j=1}^{J} ab_{i'j}$$

$$= (a_i - a_{i'}) + \sum_{j=1}^{J} (ab_{ij} - ab_{i'j})$$

[3.6] **Non-estimable** functions are:

(a) μ

(b) a_i

(c) b_j

(d) ab_{ij}

(e) $\mu + a_i$, $\mu + b_j$, $\mu + ab_{ij}$

(f) $a_i - a_{i'}$, $b_j - b_{j'}$, $ab_{ij} - ab_{i'j'}$

[4] Models with a general mean and two fixed effects, one nested within the other.

$$y_{ijk} = \mu + a_i + b_{ij} + \varepsilon_{ijk}$$

$$y_{ijk} \sim (\mu + a_i + b_{ij}, \sigma) \text{ and } \mathrm{cov}(\varepsilon_{ijk}, \varepsilon_{i'j'k'}) \neq 0 \,.$$

[4.1] $\mu - a_i + b_{ij}$ is estimable.

Proof: $E[y_{ijk}] = \mu + a_i + b_{ij}$

[4.2] $(a_i - a_{i'}) + (b_{ij} - b_{i'j})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'jk'}] \end{bmatrix} = \mu + a_i + b_{ij} - \mu - a_{i'} - b_{i'j}$$

$$= (a_i - a_{i'}) + (b_{ij} - b_{i'j})$$

[4.3] $(b_{ij} - b_{ij'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{ij'k'}] \end{bmatrix} = \mu + a_i + b_{ij} - \mu - a_i - b_{ij'}$$

$$= (b_{ij} - b_{ij'})$$

[4.4] $(a_i - a_{i'}) + (b_{ij} - b_{i'j'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'j'k'}] \end{bmatrix} = \mu + a_i + b_{ij} - \mu - a_{i'} - b_{i'j'}$$

$$= (a_i - a_{i'}) + (b_{ij} - b_{i'j'})$$

[4.5] $\sum_{i=1}^{I} (b_{ij} - b_{ij'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E\left[\sum_{i=1}^{I} y_{ijk}\right] \\ E\left[\sum_{i=1}^{I} y_{ij'k'}\right] \end{bmatrix} = \mu + \sum_{i=1}^{I} a_i + \sum_{i=1}^{I} b_{ij} - \mu - \sum_{i=1}^{I} a_i - \sum_{i=1}^{I} b_{ij'}$$

$$= \sum_{i=1}^{I} (b_{ij} - b_{ij'})$$

[4.6] **non-estimable** functions are:

(a) μ

(b) a_i

(c) b_{ij}

(d) $\mu + a_i$, $\mu + b_{ij}$

(e) $a_i - a_{i'}$, $b_{ij} - b_{i'j'}$

[5] Models with a general mean and a subclass mean.

$$y_{ijk} = \mu + c_{ij} + \varepsilon_{ijk}$$

$$y_{ijk} \sim (\mu + c_{ij}, \sigma), \text{ and } \operatorname{cov}(\varepsilon_{ijk}, \varepsilon_{i'j'k'}) \neq 0 .$$

[5.1] $\mu + c_{ij}$ is estimable.

Proof: $E[y_{ijk}] = \mu + c_{ij}$

[5.2] $(c_{ij} - c_{i'j})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{i'jk'}] \end{bmatrix} = \mu + c_{ij} - \mu - c_{i'j}$$

$$= (c_{ij} - c_{i'j})$$

[5.3] $(c_{ij} - c_{ij'})$ is estimable.

Proof:

$$[1\ -1]\begin{bmatrix} E[y_{ijk}] \\ E[y_{ij'k'}] \end{bmatrix} = \mu + c_{ij} - \mu - c_{ij'}$$

$$= (c_{ij} - c_{ij'})$$

[5.4] non-estimable functions are:

(a) μ

(b) c_{ij}

Solving the MME when the matrix X is not full column–rank

Consider the example on weaning weight analyzed earlier, but now assume that the progeny in the two herds were weaned at two different ages.

Herd	Age	Sires			Herd × age sum	Herd sum
		1	2	3		
1	1	248	296	265	809	
	2	256	282	274	812	1621
2	1	260	300	295	855	

	2	252	285	290	827	1682
Sire sum		1016	1163	1124	Total sum = 3303	

Age	1	2
Age sum	1664	1639

Let the equation for a record be:

$$y_{ijkl} = \mu + \text{herd}_i + \text{age}_j + \text{sire}_k + \text{residual}_{ijkl}.$$

In matrix notation, the corresponding linear model is:

$$y = [1 \;\; X_1 \;\; X_2] \begin{bmatrix} \mu \\ h \\ a \end{bmatrix} + Zs + e$$

or,

$$y = Xb + Zs + e$$

where

$$X = [1 \;\; X_1 \;\; X_2]$$

$$b = [\mu \;\; h \;\; a]'$$

and

$$\begin{bmatrix} y \\ \text{---} \\ s \\ e \end{bmatrix} = \left\{ \begin{bmatrix} Xb \\ \text{----} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ZGZ' + I\sigma_e^2 & | & ZG & I\sigma_e^2 \\ \text{----------} & | & \text{--} & \text{--} \\ GZ' & | & G & 0 \\ I\sigma_e^2 & | & 0 & I\sigma_e^2 \end{bmatrix} \right\}.$$

Note: If age-of-calf-at-weaning effects are present in the weaning weight records, then ($h_1° - h_2°$), the estimate of ($h_1 - h_2$) from the model without age at weaning, may be biased.

To build the MME we need:

$$[1] \qquad X'X = \begin{bmatrix} 1' \\ X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} 1 & X_1 & X_2 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1'1 & 1'X_1 & 1'X_2 \\ X_1'1 & X_1'X_1 & X_1'X_2 \\ X_2'1 & X_2'X_1 & X_2'X_2 \end{bmatrix}$$

$$X'X = \left[\begin{array}{c|cc|cc} n_{\bullet\bullet\bullet} & n_{1\bullet\bullet} & n_{2\bullet\bullet} & n_{\bullet1\bullet} & n_{\bullet2\bullet} \\ - & - & - & - & - \\ & n_{1\bullet\bullet} & 0 & n_{11\bullet} & n_{12\bullet} \\ & & n_{2\bullet\bullet} & n_{21\bullet} & n_{22\bullet} \\ - & - & - & - & - \\ \text{Sym} & & & n_{\bullet1\bullet} & 0 \\ & & & & n_{\bullet2\bullet} \end{array}\right]$$

$$X'X = \left[\begin{array}{c|cc|cc} 12 & 6 & 6 & 6 & 6 \\ - & - & - & - & - \\ & 6 & 0 & 3 & 3 \\ & & 6 & 3 & 3 \\ - & - & - & - & - \\ \mathit{Sym} & & & 6 & 0 \\ & & & & 6 \end{array}\right]$$

[2] $$X'Z = \begin{bmatrix} 1' \\ X_1' \\ X_2' \end{bmatrix} Z$$

$$X'Z = \begin{bmatrix} 1'Z \\ X_1'Z \\ X_2'Z \end{bmatrix}$$

$$X'Z = \begin{bmatrix} n_{\bullet\bullet 1} & n_{\bullet\bullet 2} & n_{\bullet\bullet 3} \\ \hline n_{1\bullet 1} & n_{1\bullet 2} & n_{1\bullet 3} \\ n_{2\bullet 1} & n_{2\bullet 2} & n_{2\bullet 3} \\ \hline n_{\bullet 11} & n_{\bullet 12} & n_{\bullet 13} \\ n_{\bullet 21} & n_{\bullet 22} & n_{\bullet 23} \end{bmatrix}$$

$$X'Z = \begin{bmatrix} 4 & 4 & 4 \\ \hline 2 & 2 & 2 \\ 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

[3] $$X'y = \begin{bmatrix} 1'y \\ X_1'y \\ X_2'y \end{bmatrix}$$

$$X'y = \begin{bmatrix} y_{\bullet\bullet\bullet\bullet} \\ ---- \\ y_{1\bullet\bullet\bullet} \\ y_{2\bullet\bullet\bullet} \\ ---- \\ y_{\bullet 1\bullet\bullet} \\ y_{\bullet 2\bullet\bullet} \end{bmatrix}$$

$$X'y = \begin{bmatrix} 3303 \\ ------ \\ 1621 \\ 1682 \\ ------ \\ 1664 \\ 1639 \end{bmatrix}$$

[4] $$Z'Z + G^{-1}\sigma_e^2 = \begin{bmatrix} n_{\bullet\bullet 1} + g^{11}\sigma_e^2 & g^{12} & g^{13} \\ g^{12} & n_{\bullet\bullet 2} + g^{22}\sigma_e^2 & g^{23} \\ g^{13} & g^{23} & n_{\bullet\bullet 3} + g^{33}\sigma_e^2 \end{bmatrix}$$

$$Z'Z + G^{-1}\sigma_e^2 = \begin{bmatrix} 4.0 + 17.78 & -3.56 & -3.56 \\ & 4.0 + 17.78 & -3.56 \\ & & 4.0 + 17.78 \end{bmatrix}$$

[5] $$Z'y = \begin{bmatrix} y_{\bullet\bullet 1\bullet} \\ y_{\bullet\bullet 2\bullet} \\ y_{\bullet\bullet 3\bullet} \end{bmatrix}$$

$$
Z'y \quad = \quad \begin{bmatrix} 1016 \\ 1163 \\ 1124 \end{bmatrix}
$$

Note: $(Z'Z + G^{-1}\sigma)$ and $Z'y$ are the same as in the model without age at weaning effects.

The MME are:

$$
\left[\begin{array}{c|cc|cc|ccc}
n_{\bullet\bullet\bullet} & n_{1\bullet\bullet} & n_{2\bullet\bullet} & n_{\bullet 1\bullet} & n_{\bullet 2\bullet} & n_{\bullet\bullet 1} & n_{\bullet\bullet 2} & n_{\bullet\bullet 3} \\
\hline
 & n_{1\bullet\bullet} & 0 & n_{11\bullet} & n_{12\bullet} & n_{1\bullet 1} & n_{1\bullet 2} & n_{1\bullet 3} \\
 & & n_{2\bullet\bullet} & n_{21\bullet} & n_{22\bullet} & n_{2\bullet 1} & 2n_{2\bullet 2} & n_{2\bullet 3} \\
\hline
 & & & n_{\bullet 1\bullet} & 0 & n_{\bullet 11} & n_{\bullet 12} & n_{\bullet 13} \\
 & & & & n_{\bullet 2\bullet} & n_{\bullet 21} & n_{\bullet 22} & n_{\bullet 23} \\
\hline
 & & & & & n_{\bullet\bullet 1} + g^{11}\sigma_e^2 & g^{12} & g^{13} \\
 & & & & & & n_{\bullet\bullet 1} + g^{22}\sigma_e^2 & g^{23} \\
 & & & & & & & n_{\bullet\bullet 1} + g^{33}\sigma_e^2
\end{array}\right]
\left[\begin{array}{c}
\mu^{\circ} \\ \hline h_1^{\circ} \\ h_2^{\circ} \\ \hline a_1^{\circ} \\ a_2^{\circ} \\ \hline \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3
\end{array}\right]
=
\left[\begin{array}{c}
y_{\bullet\bullet\bullet\bullet} \\ \hline y_{1\bullet\bullet\bullet} \\ y_{2\bullet\bullet\bullet} \\ \hline y_{\bullet 1\bullet\bullet} \\ y_{\bullet 2\bullet\bullet} \\ \hline y_{\bullet\bullet 1\bullet} \\ y_{\bullet\bullet 2\bullet} \\ y_{\bullet\bullet 3\bullet}
\end{array}\right]
$$

$$\begin{bmatrix} 12 & | & 6 & 6 & | & 6 & 6 & | & 4 & 4 & 4 \\ -- & | & -- & -- & | & -- & -- & | & ---- & ---- & ---- \\ 6 & | & 6 & 0 & | & 3 & 3 & | & 2 & 2 & 2 \\ 6 & | & 0 & 6 & | & 3 & 3 & | & 2 & 2 & 2 \\ -- & | & -- & -- & | & -- & -- & | & ---- & ---- & ----- \\ 6 & | & 3 & 3 & | & 6 & 0 & | & 2 & 2 & 2 \\ 6 & | & 3 & 3 & | & 0 & 6 & | & 2 & 2 & 2 \\ -- & | & -- & -- & | & -- & -- & | & ---- & ---- & ---- \\ 4 & | & 2 & 2 & | & 2 & 2 & | & 21.78 & -3.56 & -3.56 \\ 4 & | & 2 & 2 & | & 2 & 2 & | & -3.56 & 21.78 & -3.56 \\ 4 & | & 2 & 2 & | & 2 & 2 & | & -3.56 & -3.56 & 21.78 \end{bmatrix} \begin{bmatrix} \mu^{\circ} \\ -- \\ h_1^{\circ} \\ h_2^{\circ} \\ -- \\ a_1^{\circ} \\ a_2^{\circ} \\ -- \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} 3303 \\ ---- \\ 1621 \\ 1682 \\ ---- \\ 1664 \\ 1639 \\ --- \\ 1016 \\ 1163 \\ 1124 \end{bmatrix}$$

To solve for the vector of unknowns in the MME, first we need to know the rank of the LHS. **If the LHS is not full rank we will need to put constraints on the solutions to solve the MME.**

Notice that:

$$\begin{aligned} rank\begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z + G^{-1}\sigma_e^2 \end{bmatrix} &= rank\begin{bmatrix} X'X \\ Z'X \end{bmatrix} + rank\begin{bmatrix} X'Z \\ Z'Z + G^{-1}\sigma_e^2 \end{bmatrix} \\ &= rank\left[X'X\right] + rank\left[Z'Z + G^{-1}\sigma_e^2\right] \\ &= rank\left[X\right] + rank\left[Z'Z + G^{-1}\sigma_e^2\right]. \end{aligned}$$

But $\left[Z'Z + G^{-1}\sigma_e^2\right]$ is always full rank, thus to find the rank of the LHS we only need to determine the rank of $\left[X'X\right]$.

$$\text{rank}\left[\, X'X \,\right] = \text{rank}\begin{bmatrix} 12 & | & 6 & 6 & | & 6 & 6 \\ -- & | & -- & -- & | & -- & -- \\ 6 & | & 6 & 0 & | & 3 & 3 \\ 6 & | & 0 & 6 & | & 3 & 3 \\ -- & | & -- & -- & | & -- & -- \\ 6 & | & 3 & 3 & | & 6 & 0 \\ 6 & | & 3 & 3 & | & 0 & 6 \end{bmatrix} = 3,$$

i.e., there are only 3 independent columns (or rows) in [X′X], because:

(1) the first column is equal to the sum of the second and third columns, and

(2) the sum of the second and third columns is equal to the sum of the fourth and fifth columns.

Thus, we **need to impose constraints on the solutions** to solve for the fixed effects in the MME.

Constraints on the solutions

[1] **Set some solutions equal to zero.** This implies that the columns and rows of the LHS and the element of the RHS corresponding to these solutions are zeroed out.

$$\begin{bmatrix} \textit{Number of solutions} \\ \textit{set equal to zero} \end{bmatrix} = \begin{bmatrix} \textit{Order of} \\ [\ X'X\ ] \end{bmatrix} - \begin{bmatrix} \textit{Rank of} \\ [\ X'X\ ] \end{bmatrix}$$

Also, the solutions set to zero are chosen such that the

$$\begin{bmatrix} \textit{Number of equations} \\ \textit{left in}\,[\ X'X\ ] \end{bmatrix} = \begin{bmatrix} \textit{Rank of} \\ [\ X'X\ ] \end{bmatrix}$$

For instance, set $\boldsymbol{\mu}° = \mathbf{h}_1° = \mathbf{0}$ in the example.

Then, the **vector of solutions** is equal to:

$$\begin{bmatrix} \mu^\circ \\ \hline h_1^\circ \\ h_2^\circ \\ \hline a_1^\circ \\ a_2^\circ \\ \hline \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \left[\begin{array}{c|cc|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3 & 3 & 2 & 2 & 2 \\ \hline 0 & 0 & 3 & 6 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 0 & 6 & 2 & 2 & 2 \\ \hline 0 & 0 & 2 & 2 & 2 & 21.78 & -3.56 & -3.56 \\ 0 & 0 & 2 & 2 & 2 & -3.56 & 21.78 & -3.56 \\ 0 & 0 & 2 & 2 & 2 & -3.56 & -3.56 & 21.78 \end{array}\right]^{-} \begin{bmatrix} 0 \\ \hline 0 \\ 1682 \\ \hline 1664 \\ 1639 \\ \hline 1016 \\ 1163 \\ 1124 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \\ 10.1667 \\ \hline 272.2500 \\ 268.0833 \\ \hline -3.3553 \\ 2.4474 \\ 0.9079 \end{bmatrix}$$

$$\text{Let } K' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} k_1' \\ k_2' \\ k_3' \\ k_4' \end{bmatrix}$$

Are the $\{k_i'b\}$ estimable?

$k_i'b$ is estimable $\Leftrightarrow$ $k_i'[B_{11}X'X + B_{12}Z'X] = k_i'$, where B_{11} and B_{12} are submatrices of the inverse of the LHS, i.e.,

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix} = \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z + G^{-1}\sigma_e^2 \end{bmatrix}^{-}.$$

Thus,

$K'[B_{11}X'X + B_{12}Z'X]$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & -0.17 \\ 0 & 0 & -0.17 & 0.28 & 0.12 \\ 0 & 0 & -0.17 & 0.12 & 0.28 \end{bmatrix} \begin{bmatrix} 12 & 6 & 6 & 6 & 6 \\ 6 & 6 & 0 & 3 & 3 \\ 6 & 0 & 6 & 3 & 3 \\ 6 & 3 & 3 & 6 & 0 \\ 6 & 3 & 3 & 0 & 6 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.03 & -0.03 & -0.03 \\ -0.03 & -0.03 & -0.03 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix}$$

$\Rightarrow$ $k_2'b$ and $k_3'b$ are estimable, and

$\Rightarrow$ $k_1'b$ and $k_4'b$ are not estimable:

(1) $k_1'b^\circ$ estimates nothing, and

(2) $k_4'b^\circ$ is an estimate of $-(\mu + h_1 + a_1)$.

The estimates of $k_2'b$ and $k_3'b$ are:

$$\begin{bmatrix} k_2'b^\circ \\ k_3'b^\circ \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10.17 \\ 272.25 \\ 268.08 \end{bmatrix} = \begin{bmatrix} -10.17 \\ 4.17 \end{bmatrix}.$$

[2] Set some solutions, or the sum of some solutions, to zero using Lagrange Multipliers.

Here, equations are added to the MME, thus the order of the MME is larger than in procedure [1].

The MME with Lagrange multipliers is:

$$\begin{bmatrix} X'X & X'Z & L \\ Z'X & Z'Z + G^{-1}\sigma_e^2 & 0 \\ L' & 0 & 0 \end{bmatrix} \begin{bmatrix} h^\circ \\ \hat{s} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} X'y \\ Z'y \\ 0 \end{bmatrix}.$$

If μ° and h were set to zero the matrix L' would be:

$$L' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and if $\sum_{i=1}^{2} h_i{}^\circ = \sum_{j=1}^{2} a_j{}^\circ = 0$, then L' would be:

$$L' = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

[2.1] The **solution vector** of the MME, for the constraint $\boldsymbol{\mu^\circ = h_1{}^o = 0}$, is equal to:

$$\begin{bmatrix} \mu^\circ \\ --- \\ h_1^\circ \\ h_2^\circ \\ --- \\ a_1^\circ \\ a_2^\circ \\ --- \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ --- \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 12 & | & 6 & 6 & | & 6 & 6 & | & 4 & 4 & 4 & | & 1 & 0 \\ -- & | & -- & -- & | & -- & -- & | & ----- & ----- & ----- & | & -- & -- \\ 6 & | & 6 & 0 & | & 3 & 3 & | & 2 & 2 & 2 & | & 0 & 1 \\ 6 & | & 0 & 6 & | & 3 & 3 & | & 2 & 2 & 2 & | & 0 & 0 \\ -- & | & -- & -- & | & -- & -- & | & ----- & ----- & ----- & | & -- & -- \\ 6 & | & 3 & 3 & | & 6 & 0 & | & 2 & 2 & 2 & | & 0 & 0 \\ 6 & | & 3 & 3 & | & 0 & 6 & | & 2 & 2 & 2 & | & 0 & 0 \\ -- & | & -- & -- & | & -- & -- & | & ----- & ----- & ----- & | & -- & -- \\ 4 & | & 2 & 2 & | & 2 & 2 & | & 21.78 & -3.56 & -3.56 & | & 0 & 0 \\ 4 & | & 2 & 2 & | & 2 & 2 & | & -3.56 & 21.78 & -3.56 & | & 0 & 0 \\ 4 & | & 2 & 2 & | & 2 & 2 & | & -3.56 & -3.56 & 21.78 & | & 0 & 0 \\ -- & | & -- & -- & | & -- & -- & | & ----- & ----- & ----- & | & -- & -- \\ 1 & | & 0 & 0 & | & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 \\ 0 & | & 1 & 0 & | & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3303 \\ ---- \\ 1621 \\ 1682 \\ ---- \\ 1664 \\ 1639 \\ ---- \\ 1016 \\ 1163 \\ 1124 \\ ---- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ ------- \\ 0 \\ 10.1667 \\ -------- \\ 272.2500 \\ 268.0833 \\ -------- \\ -3.553 \\ 2.4474 \\ 0.9079 \\ -------- \\ 0 \\ 0 \end{bmatrix}$$

$$K'[B_{11}X'X + B_{12}Z'X] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix}$$

$$\neq K'$$

⇒ $k_2'b$ and $k_3'b$ are estimable, but $k_1'b$ and $k_4'b$ are not. Again, $k_1'b^\circ$ estimates nothing, and $k_4'b^\circ$ is an estimate of $-(\mu + h_1 + a_1)$.

The values of the estimable functions are:

$$\begin{bmatrix} k_2'b^\circ \\ k_3'b^\circ \end{bmatrix} = \begin{bmatrix} 0 & -10.17 \\ 272.25 & -268.08 \end{bmatrix} = \begin{bmatrix} -10.17 \\ 4.17 \end{bmatrix}.$$

Note that the same solutions for b were obtained by setting μ° and h directly and through Lagrange multipliers.

[2.2] The **solution vector** of the MME, for the constraint $\sum_{i=1}^{2} h_i{}^\circ = \sum_{j=1}^{2} a_j{}^\circ = 0$, is equal to:

$$\begin{bmatrix} \mu^\circ \\ --- \\ h_1{}^\circ \\ h_2{}^\circ \\ --- \\ a_1{}^\circ \\ a_2{}^\circ \\ --- \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ --- \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} = \left[\begin{array}{c|cc|cc|ccc|cc} 12 & 6 & 6 & 6 & 6 & 4 & 4 & 4 & 0 & 0 \\ \hline 6 & 6 & 0 & 3 & 3 & 2 & 2 & 2 & 1 & 0 \\ 6 & 0 & 6 & 3 & 3 & 2 & 2 & 2 & 1 & 0 \\ \hline 6 & 3 & 3 & 6 & 0 & 2 & 2 & 2 & 0 & 1 \\ 6 & 3 & 3 & 0 & 6 & 2 & 2 & 2 & 0 & 1 \\ \hline 4 & 2 & 2 & 2 & 2 & 21.78 & -3.56 & -3.56 & 0 & 0 \\ 4 & 2 & 2 & 2 & 2 & -3.56 & 21.78 & -3.56 & 0 & 0 \\ 4 & 2 & 2 & 2 & 2 & -3.56 & -3.56 & 21.78 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^{-1} \begin{bmatrix} 3303 \\ ---- \\ 1621 \\ 1682 \\ ---- \\ 1664 \\ 1639 \\ ---- \\ 1016 \\ 1163 \\ 1124 \\ ---- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 275.2500 \\ ------- \\ -5.0833 \\ 5.0833 \\ -------- \\ 2.0833 \\ -2.0833 \\ -------- \\ -3.3553 \\ 2.4474 \\ 0.9079 \\ -------- \\ 0 \\ 0 \end{bmatrix}$$

$$K'[B_{11}X'X \mid B_{12}Z'X] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

⇒ $k_2'b$ and $k_3'b$ are estimable, and

⇒ $k_1'b$ and $k_4'b$ are not estimable:

(1) $k_1'b°$ is an estimate of $[\mu + 0.5(h_1 + h_2) + 0.5(a_1 + a_2)]$, and

(2) $k_4'b°$ is an estimate of $[0.5(h_1 - h_2) + 0.5(a_2 - a_1)]$.

The values of the estimable functions $k_2'b°$ and $k_3'b°$ are:

$$\begin{bmatrix} k_2'b° \\ k_3'b° \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 275.250 \\ -5.083 \\ 5.083 \\ 2.083 \\ -2.083 \end{bmatrix} = \begin{bmatrix} -10.17 \\ 4.17 \end{bmatrix}.$$

Remarks

[1] The estimable function $(h_1 - h_2)$ had the same value in models 1 (without age at weaning effects) and model 2 (with age at weaning effects) because the bias due to ignoring age at weaning effects in model 1 was zero, i.e., $[1\ -1]\ [B_{11}X_1'X_2] = 0$, where B_{11} belongs to $(LHS)^-$ for model 1.

Proof that ($h_1^\circ - h_2^\circ$) from model 1 is unbiased.

$$[B_{11}X_1'X_2] = \begin{bmatrix} 0.197917 & 0.03125 \\ 0.03125 & 0.197917 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0.68751 & 0.68751 \\ 0.68751 & 0.68751 \end{bmatrix}$$

$$\Rightarrow \quad k_2'[B_{11}X_1'X_2] = [1\ -1]\begin{bmatrix} 0.68751 & 0.68751 \\ 0.68751 & 0.68751 \end{bmatrix}$$

$$= [0]$$

$\Rightarrow$ ($h_1^\circ - h_2^\circ$) is an unbiased estimate of ($h_1 - h_2$) in this particular case.

$$\Rightarrow \quad \text{MSE}\,(h_1^\circ - h_2^\circ) = \text{var}(h_1^\circ - h_2^\circ) + [\text{bias}\,(h_1^\circ - h_2^\circ)]^2$$

$$= \text{var}(h_1^\circ - h_2^\circ) + 0$$

$$= \text{var}(h_1^\circ - h_2^\circ)$$

[2] The BLUP of s, $\hat{s}$, was the same in model 1 (without age at weaning effects) and in model 2 (with age at weaning effects). This occurred because the bias in model 1, due to ignoring age at weaning effects, was zero, i.e.,

$$B_{12}'X_1'X_2 + B_{22}Z'X_2 = 0,$$

where B_{12}' and B_{22} are submatrices of the $(\text{LHS})^-$ from model 1.

Proof that the $\{\hat{s}_i\}$ from model 1 are unbiased

$[B_{12}'X_1'X_2 + B_{22}Z'X_2]$

$$= \left\{ \begin{bmatrix} -0.03125 & -0.03125 \\ -0.03125 & -0.03125 \\ -0.03125 & -0.03125 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 0.0576 & 0.01809 & 0.01809 \\ 0.01809 & 0.0576 & 0.01809 \\ 0.01809 & 0.01809 & 0.0576 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} -0.1875 & -0.1875 \\ -0.1875 & -0.1875 \\ -0.1875 & -0.1875 \end{bmatrix} + \begin{bmatrix} 0.1875 & 0.1875 \\ 0.1875 & 0.1875 \\ 0.1875 & 0.1875 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ the $\{s_i\}$ are unbiased.

Summary Table

	$(h_1^\circ - h_2^\circ)$	$SE(h_1^\circ - h_2^\circ))$	$a_1^\circ - a_2^\circ$	$SE(a_1^\circ - a_2^\circ)$
$\mu^\circ = h_1^\circ = 0$	−10.17	5.38	4.17	5.38
Lagrange: $\mu^\circ = h_1^\circ = 0$	−10.17	5.38	4.17	5.38
$\sum_{i=1}^{2} h_i^\circ = \sum_{j=1}^{2} a_j^\circ = 0$	−10.17	5.38	4.17	5.38

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