# ANIMAL BREEDING NOTES

## CHAPTER 16

## ANIMAL AND REDUCED ANIMAL MODELS

### Animal Model (AM)

**Objective:** to predict the breeding value (BV) of animals based on their own records and(or) records of their relatives.

**Assumptions:**

(i) Animals belong to a single population,

(ii) Animals may have 1 or more records and covariances among records are due only to genetic factors, and

(iii) There is either no selection in the population, or:

(a) If selection occurred based on records, the selection was within fixed effects, and

(b) If selection occurred based on the BV of animals, the relationship matrix is complete.

The **AM** is:

$$y = Xb + Zu + e$$

$$E[y] = Xb$$

$$\text{var}\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}$$

$$= \begin{bmatrix} A\sigma_A^2 & 0 \\ 0 & I\sigma_e^2 \end{bmatrix}, \text{ because of assumptions (i) and (ii).}$$

$$\Rightarrow \quad \text{var}(y) = ZGZ' + R,$$

$$= ZAZ'\sigma_A^2 + I\sigma_e^2,$$

where

$y$ = vector of animal records,

$b$ = vector of unknown fixed effects,

$u$ = vector of unknown random BV belonging to the animals making the records,

$e$ = vector of unknown random residual effects,

$X$ = known incidence matrix relating records to fixed effects in vector b,

$Z$ = known incidence matrix relating records to BV in vector u.

Let

$$\alpha = \frac{\sigma_A^2}{\sigma_e^2}.$$

Then, the mixed model equations (MME) for the **AM**, **after multiplying both sides by $\sigma_e^2$**, are:

$$\begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z + A^{-1}\alpha^{-1} \end{bmatrix} \begin{bmatrix} b \\ u \end{bmatrix} = \begin{bmatrix} X'y \\ Z'y \end{bmatrix}$$

where $A^{-1}$ is the inverse of the matrix of additive relationships among the animals with records. If animals in the pedigree of animals with records did **not** have records themselves, their BV would not be included in vector u. This would prevent the use of Henderson's rules to compute $A^{-1}$ directly. However, **Henderson's rules could be used if we included the BV of the animals without records in vector u**, which can be accomplished by using the following **Equivalent Animal Model (EAM)**:

$$y = Xb + \begin{bmatrix} 0 & Z \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + e$$

$$E[y] = Xb$$

$$\text{var}\begin{bmatrix} u_0 \\ u_1 \\ e \end{bmatrix} = \begin{bmatrix} A_{00}\sigma_A^2 & A_{01}\sigma_A^2 & 0 \\ A_{10}\sigma_A^2 & A_{11}\sigma_A^2 & 0 \\ 0 & 0 & I\sigma_e^2 \end{bmatrix}$$

where

$u_1$ = u of the **AM**,

$u_0$ = random vector of the BV of animals without records which are relatives of the animals with records.

The variance of y is:

$$\text{var(y)} = \begin{bmatrix} 0 & Z \end{bmatrix} \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} 0 \\ Z' \end{bmatrix} \sigma_A^2 + I\sigma_e^2$$

$$= Z A_{11} Z' \sigma_A^2 + I\sigma_e^2$$

Thus, **the E[y] and the var(y) of the AM and the EAM are the same, proving that they are equivalent models.**

The MME of the **EAM** are:

$$\begin{bmatrix} X'X & 0 & X'Z \\ 0 & A^{00}\alpha^{-1} & A^{01}\alpha^{-1} \\ Z'X & A^{10}\alpha^{-1} & Z'Z + A^{11}\alpha^{-1} \end{bmatrix} \begin{bmatrix} b \\ u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} X'y \\ 0 \\ Z'y \end{bmatrix}$$

where

$$\begin{bmatrix} A^{00} & A^{01} \\ A^{10} & A^{11} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}^{-1}$$, computed using Henderson's rules.

**Remarks:**

(1) Absorption of the equations for $u_0$ into b and $u_1$ yields:

$$\begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z + (A^{11} - A^{10}(A^{00})^{-1}A^{01})\alpha^{-1} \end{bmatrix} \begin{bmatrix} b \\ u_1 \end{bmatrix} = \begin{bmatrix} X'y \\ Z'y \end{bmatrix}$$

where

$$\begin{aligned} (A^{11} - A^{10}(A^{00})^{-1}A^{01})\alpha^{-1} &= A_{11}^{-1}\alpha^{-1} \\ &= A^{-1}\alpha^{-1} \end{aligned}$$

**Proof:**

$$\begin{bmatrix} A^{00} & A^{01} \\ A^{10} & A^{11} \end{bmatrix} \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$A^{00}A_{01} + A^{01}A_{11} = 0 \qquad [1]$$

$$A^{10}A_{01} + A^{11}A_{11} = I \qquad [2]$$

From [1]:

$$A_{01} = -(A^{00})^{-1}A^{01}A_{11} \qquad [3]$$

Substituting [3] for $A_{01}$ in [2] yields:

$$\begin{aligned} -A^{10}(A^{00})^{-1}A^{01}A_{11} + A^{11}A_{11} &= I \\ (A^{11} - A^{10}(A^{00})^{-1}A^{01})A_{11} &= I \end{aligned}$$

$\Rightarrow$ $A_{11}^{-1} = (A^{11} - A^{10}(A^{00})^{-1}A^{01})$

$\Rightarrow$ $K'b^{\circ}$ ε AM = $K'b^{\circ}$ ε EAM for estimable $K'$

$\Rightarrow$ $\hat{u}$ from AM = $\hat{u}_1$ from EAM

(2) From the equations for $u_0$ of the MME for the **EAM**,

$$A^{00}\alpha^{-1}\hat{u}_0 = -A^{01}\alpha^{-1}\hat{u}_1$$

$$\Rightarrow \qquad \hat{u}_0 = -(A^{00})^{-1}A^{01}\hat{u}_1$$

or

$$\hat{u}_0 = -(A^{00})^{-1} A^{01} \hat{u}$$

where

$$\hat{u}_1 = \text{BLUP of } u_1 \ (= u) \text{ from the } \textbf{EAM}, \text{ and}$$

$$\hat{u} = \text{BLUP of } u \ (= u_1) \text{ from the } \textbf{AM}.$$

Thus, $\hat{u}_0$ is the BLUP of $u_0$, because $u_0$ is a linear combination of $\hat{u}_1 = \hat{u}$ , the BLUP of u.

**Also, notice that:**

$$\hat{u}_0 = -(A^{00})^{-1} A^{01} \hat{u}_1$$

$$= A_{01} A_{11}^{-1} \hat{u}_1$$

**Proof:**

Equation [1] above is:

$$A^{00} A_{01} + A^{01} A_{11} = 0$$

$$\Rightarrow \quad A^{01} = -A^{00} A_{01} A$$

Substituting $-A^{00}A_{01}$ A for $A^{01}$ in $\hat{u}_0 = -(A^{00})^{-1} A^{01} \hat{u}_1$ yields:

$$\hat{u}_0 = -(A^{00})^{-1} (-A^{00}) A_{01} A_{11}^{-1} \hat{u}_1$$

$$\Rightarrow \quad \hat{u}_0 = A_{01} A_{11}^{-1} \hat{u}_1$$

or

$$\hat{u}_0 = A_{01} A_{11}^{-1} \hat{u}$$

(3) This method used to obtain the BLUP of $u_0$ is actually a **general method to predict the BV of animals not represented in vector u, but correlated to some of the elements of u** (see Henderson, 1977).

(4) Advantages of the **EAM**:

(a) all additive genetic relationships are used, and

(b) the MME are easy to construct.

(5) Disadvantages of the **EAM**:

(a) there are usually more equations (i.e., more unknowns) than there are records, and

(b) sometimes the MME do not behave well in iterative solutions (probably as a consequence of disadvantage 5a).

## Reduced Animal Model (RAM)

**Objective:** same as the **AM** (or the **EAM**), i.e., to predict the BV of animals with and without records. However, the set of equations used to compute the BV will require less number of computations. The reduction in computations is achieved by exploiting the structure of A.

**Assumptions:** same as the **AM**.

### Derivation of the RAM

Denote the **EAM**:

$$y = Xb + \begin{bmatrix} 0 & Z \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + e \qquad [4]$$

$$y \sim \left( Xb,\ Z A_{11} Z' \sigma_A^2 + I \sigma_e^2 \right)$$

as:

$$y = Xb + \dot{Z}\dot{u} + e \qquad [5]$$

$$y \sim \left( Xb,\ \dot{Z} A_{11} \dot{Z}' \sigma_A^2 + I \sigma_e^2 \right)$$

where

$$\dot{Z} = [0 \quad Z]$$

$$\dot{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Consider ordering the y and the $\dot{u}$ vectors of the **EAM** as follows:

$$y = \begin{bmatrix} y_p \\ y_n \end{bmatrix}, \text{ and } \dot{u} = \begin{bmatrix} u_p \\ u_n \end{bmatrix},$$

where

$y_p$ = subvector of y containing records of animals that have progeny with records,

$y_n$ = subvector of y with records of animals that are nonparents,

$u_p$ = subvector of $\dot{u}$ representing the BV of parents with or without records of their own, and

$u_n$ = subvector of $\dot{u}$ holding the BV of nonparents.

The **EAM** corresponding this partitioning of the y and $\dot{u}$ is:

$$\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b + \begin{bmatrix} Z_p & 0 \\ 0 & Z_n \end{bmatrix} \begin{bmatrix} u_p \\ u_n \end{bmatrix} + \begin{bmatrix} e_p \\ e_n \end{bmatrix} \qquad [6]$$

$$E\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b$$

$$\text{var}\begin{bmatrix} u_p \\ u_n \\ ---- \\ e_p \\ e_n \end{bmatrix} = \begin{bmatrix} A_{pp}\alpha & A_{pn}\alpha & | & 0 & 0 \\ A_{np}\alpha & A_{nn}\alpha & | & 0 & 0 \\ ---- & ---- & | & --- & --- \\ 0 & 0 & | & I_p & 0 \\ 0 & 0 & | & 0 & I_n \end{bmatrix} \sigma_e^2$$

$$\Rightarrow \qquad \text{var}\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} Z_p & 0 \\ 0 & Z_n \end{bmatrix} \begin{bmatrix} A_{pp} & A_{pn} \\ A_{np} & A_{nn} \end{bmatrix} \begin{bmatrix} Z_p' & 0 \\ 0 & Z_n \end{bmatrix} \sigma_A^2 + \begin{bmatrix} I_p & 0 \\ 0 & I_n \end{bmatrix} \sigma_e^2$$

$$= \mathrm{Z\,A\,Z'}\sigma_A^2 + \mathrm{I}\,\sigma_e^2$$

$$= \mathrm{var}(\mathrm{y})$$

$\Rightarrow$ the **EAM** [5] and [6] are equivalent models.

The MME for the **EAM** [6] are:

$$\begin{bmatrix} X_p'X_p + X_n'X_n & X_p'Z_p & X_n'Z_n \\ Z_p'X_p & Z_p'Z_p + A^{pp}\alpha^{-1} & -A^{pn}\alpha^{-1} \\ Z_n'X_n & -A^{np}\alpha^{-1} & Z_n'Z_n + A^{nn}\alpha^{-1} \end{bmatrix} \begin{bmatrix} b \\ u_p \\ u_n \end{bmatrix} = \begin{bmatrix} X_p'y_p + X_n'y_n \\ Z_p'y_p \\ Z_n'y_n \end{bmatrix} \qquad [7]$$

where

$$\begin{bmatrix} A^{pp} & A^{pn} \\ A^{np} & A^{nn} \end{bmatrix} = \begin{bmatrix} A_{pp} & A_{pn} \\ A_{np} & A_{nn} \end{bmatrix}^{-1} = A^{-1}.$$

However,

$$A^{-1} = (I - \tfrac{1}{2}P')\, D^{-1}\, (I - \tfrac{1}{2}P)$$

where

$$P = \begin{bmatrix} P_{pp} & 0 \\ P_{np} & P_{nn} \end{bmatrix},\ D^{-1} = \begin{bmatrix} D_p^{-1} & 0 \\ 0 & D_n^{-1} \end{bmatrix},\ I = \begin{bmatrix} I_p & 0 \\ 0 & I_n \end{bmatrix},$$

**But $P_{nn} = 0$, because animals in $u_n$ have no progeny**.

Thus,

$$P = \begin{bmatrix} P_{pp} & 0 \\ P_{np} & 0 \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} (I - \tfrac{1}{2}P_{pp}') & -\tfrac{1}{2}P_{np}' \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D_p^{-1} & 0 \\ 0 & D_n^{-1} \end{bmatrix} \begin{bmatrix} (I_p - \tfrac{1}{2}P_{pp}) & 0 \\ -\tfrac{1}{2}P_{np} & I_n \end{bmatrix}$$

$$= \begin{bmatrix} (I_p - ½ P_{pp}')D_p^{-1}(I_p - ½ P_{pp}) + (½ P_{np}')D_n^{-1}(½ P_{np}) & | & -(½ P_{np}')D_n^{-1} \\ -D_n^{-1}(½ P_{np}) & | & D_n^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A_{pp}^{-1} + ¼ P_{np}' D_n^{-1} P_{np} & | & -½ P_{np}' D_n^{-1} \\ -½ D_n^{-1} P_{np} & | & D_n^{-1} \end{bmatrix} \quad [8]$$

$$= \begin{bmatrix} A^{pp} & A^{pn} \\ A^{np} & A^{nn} \end{bmatrix}$$

Substituting [8] for the submatrices of the $A^{-1}$ of MME [7] yields:

$$\begin{bmatrix} X_p'X_p + X_n'X_n & X_p'Z_p & X_n'Z_n \\ Z_p'X_p & Z_p'Z_p + A_{pp}^{-1}\alpha^{-1} + ¼ P_{np}'D_n^{-1}P_{np}\alpha^{-1} & -½ P_{np}'D_n^{-1}\alpha^{-1} \\ Z_n'X_n & -½ D_n^{-1}P_{np}\alpha^{-1} & Z_n'Z_n + D_n^{-1}\alpha^{-1} \end{bmatrix} \begin{bmatrix} b \\ u_p \\ u_n \end{bmatrix} = \begin{bmatrix} X_p'y_p + X_n'y_n \\ Z_p'y_p \\ Z_n'y_n \end{bmatrix} \quad [9]$$

**Remarks:**

(a) The BLUP of $u_n$, i.e., $\hat{u}_n$, can be easily computed based on b° and $\hat{u}_p$, i.e., from the third equation of [9],

$$\hat{u}_n = (Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(Z_n'y_n - Z_n'X_nb^\circ - ½\alpha^{-1}D_n^{-1}P_{np}\hat{u}_p)$$

$$\hat{u}_n = (Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(Z_n'y_n - Z_n'X_nb^\circ) \quad \} \text{ data}$$

$$+ (Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(-½\alpha^{-1}D_n^{-1}P_{np}\hat{u}_p) \quad \} \text{ pedigree}$$

The BLUP of the BV of the $i^{th}$ nonparent is:

$$\hat{u}_{n_i} = \left(n_{\bullet i} + d_{ii}^{-1}\alpha^{-1}\right)^{-1}\left(y_{\bullet i \bullet} - \sum_k n_{ki} b_k^\circ\right) + \left(n_{\bullet i} + d_{ii}^{-1}\alpha^{-1}\right)^{-1}\left[-½\alpha^{-1}d_{ii}^{-1}\left(\delta_{s_i}\hat{u}_{s_i} + \delta_{d_i}\hat{u}_{d_i}\right)\right]$$

where $\delta_{s_i}$ and $\delta_{d_i}$ are Kronecker deltas,

$$\left(n_{\bullet i} + d_{ii}^{-1}\,\alpha^{-1}\right)^{-1} = \frac{1}{n_{\bullet i} + \dfrac{\alpha^{-1}}{d_{ii}}}$$

$$= \frac{d_{ii}}{n_{\bullet i}\, d_{ii} + \alpha^{-1}}$$

and

$$\left(n_{\bullet i} + d_{ii}^{-1}\,\alpha^{-1}\right)^{-1}\left(\alpha^{-1}\, d_{ii}^{-1}\right) = \frac{\left(\dfrac{\alpha^{-1}}{d_{ii}}\right)}{\left(\dfrac{n_{\bullet i}\, d_{ii} + \alpha^{-1}}{d_{ii}}\right)}$$

$$= \frac{\alpha^{-1}}{n_{\bullet i}\, d_{ii} + \alpha^{-1}}$$

Thus,

$$\hat{u}_{n_i} = \frac{d_{ii}}{n_{\bullet i}\, d_{ii} + \alpha^{-1}}\left[ y_{\bullet i \bullet} - \sum_k n_{ki}\, b_k^{\circ} \right] + \frac{\alpha^{-1}}{n_{\bullet i}\, d_{ii} + \alpha^{-1}}\left[ -½\left( \delta_{s_i}\, \hat{u}_{s_i} + \delta_{d_i}\, \hat{u}_{d_i} \right)\right]$$

If nonparents have only one record each, $n_{\bullet i} = 1$, then

$$\hat{u}_{n_i} = \frac{d_{ii}}{d_{ii} + \alpha^{-1}}\left[\, y_i - b_i^{\circ} \,\right] + \frac{\alpha^{-1}}{d_{ii} + \alpha^{-1}}\left[ -½\left(\delta_{s_i}\, \hat{u}_{s_i} + \delta_{d_i}\, \hat{u}_{d_i}\right) \right]$$

Let

$$w_n^{ii} = \frac{d_{ii}}{d_{ii} + \alpha^{-1}}$$

Note that,

$$1 - w_n^{ii} = \left( \frac{d_{ii} + \alpha^{-1}}{d_{ii} + \alpha^{-1}} \right) - \left( \frac{d_{ii}}{d_{ii} + \alpha^{-1}} \right)$$

$$= \frac{\alpha^{-1}}{d_{ii} + \alpha^{-1}}$$

Thus,

$$\hat{u}_{n_i} = w_n^{ii}\left( y_i - b_i{}^\circ\right) + \left(1 - w_n^{ii}\right)\left[-½\left(\delta_{s_i}\hat{u}_{s_i} + \delta_{d_i}\hat{u}_{d_i}\right)\right]$$

(b) The matrix $Z_n'Z_n + D_n^{-1}\alpha^{-1}$ is diagonal. Thus, the **equations for $u_n$ can be readily absorbed into b and $u_p$** (**Hint: obtain $(\alpha Z_n D_n Z_n' + I)^{-1}$ in all equations**). The absorption of $u_n$ into b and $u_p$ is as follows:

(i) $$X_p'X_p + X_n'X_n - X_n'Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n'X_n$$

$$= X_p'X_p + X_n'[I - Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n']X_n$$

$$= X_p'X_p + X_n'(\alpha Z_n D_n Z_n' + I)^{-1}X_n$$

(ii) $$X_p'Z_p - X_n'Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(-½D_n^{-1}P_{np}\alpha^{-1})$$

$$= X_p'Z_p + X_n'Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(-D_n^{-1}\alpha^{-1} + Z_n'Z_n - Z_n'Z_n)(½P_{np})$$

$$= X_p'Z_p + X_n'Z_n[I - (Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n'Z_n](½P_{np})$$

$$= X_p'Z_p + X_n'[I - Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n'](½Z_nP_{np})$$

$$= X_p'Z_p + X_n'(\alpha Z_n D_n Z_n' + I)^{-1}(½Z_nP_{np})$$

(iii) $$Z_p'Z_p + A\alpha^{-1} + (½P_{np}')(D_n^{-1}\alpha^{-1})(½P_{np})$$

$$- (½P_{np}')(D_n^{-1}\alpha^{-1})(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}(D_n^{-1}\alpha^{-1})(½P_{np})$$

$$= Z_p'Z_p + A\alpha^{-1} + (½P_{np}')(D_n^{-1}\alpha^{-1})[D_n\alpha - (ZZ_n + D_n^{-1}\alpha^{-1})^{-1}](D_n^{-1}\alpha^{-1})(½P_{np})$$

$$= Z_p'Z_p + A\alpha^{-1} + (½P_{np}')(D_n^{-1}\alpha^{-1}$$

$$+ Z_n'Z_n - Z_n'Z_n)[D_n\alpha - (Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}](D_n^{-1}\alpha^{-1})(½P_{np})$$

$$= Z_p'Z_p + A\alpha^{-1} + (½P_{np}')[I - I + Z_n'Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}]$$

$$(D_n^{-1}\alpha^{-1} + Z_n'Z_n - Z_n'Z_n)(½P_{np})$$

$$= Z_p'Z_p + A\alpha^{-1} + (½P_{np}')[Z_n'Z_n - Z_n'Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n'Z_n](½P_{np})$$

$$= Z_p'Z_p + A\alpha^{-1} + (½P_{np}'Z_n')[I - Z_n(Z_n'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n'](½Z_nP_{np})$$

$$= \quad Z_p{}'Z_p + A\alpha^{-1} + (½P_{np}{}'Z_n{}')(\alpha Z_n D_n Z_n{}' + I)^{-1}(½Z_n P_{np})$$

(iv) $$X_p{}'y_p + X_n{}'y_n - X_n{}'Z_n(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}'y_n$$

$$= \quad X_p{}'y_p + X_n{}'[I - Z_n(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}']y_n$$

$$= \quad X_p{}'y_p + X_n{}'(\alpha Z_n D_n Z_n{}' + I)^{-1}y_n$$

(v) $$Z_p{}'y_p - (½P_{np}{}')(D_n^{-1}\alpha^{-1})(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}'y_n$$

$$= \quad Z_p{}'y_p - (½P_{np}{}')(-Z_n{}'Z_n + Z_n{}'Z_n + D_n^{-1}\alpha^{-1})(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}'y_n$$

$$= \quad Z_p{}'y_p + (½P_{np}{}')[Z_n{}'Z_n(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}' + Z_n{}']y_n$$

$$= \quad Z_p{}'y_p + (½P_{np}{}'Z_n{}')[I + Z_n(Z_n{}'Z_n + D_n^{-1}\alpha^{-1})^{-1}Z_n{}']y_n$$

$$= \quad Z_p{}'y_p + (½P_{np}{}'Z_n{}')(\alpha Z_n D_n Z_n{}' + I)^{-1}y_n$$

Let $\mathbf{R_2} = (\alpha Z_n D_n Z_n{}' + I)$

Then, **the MME after absorbing $\mathbf{u_n}$** are:

$$\begin{bmatrix} X_p{}'X_p + X_n{}'R_2^{-1}X_n & X_p{}'Z_p + X_n{}'R_2^{-1}Z_n P_{np}(½) \\ Z_p{}'X_p + (½)P_{np}{}'Z_n{}'R_2^{-1}X_n & Z_p{}'Z_p + A_{pp}^{-1}\alpha^{-1} + (½)P_{np}{}'Z_n{}'R_2^{-1}Z_n P_{np}{}'(½) \end{bmatrix} \begin{bmatrix} b^\circ \\ \hat{u}_p \end{bmatrix} = \begin{bmatrix} X_p{}'y_p + X_n{}'R_2^{-1}y_n \\ Z_p{}'y_p + (½)P_{np}{}'Z_n{}'R_2^{-1}y_n \end{bmatrix} \quad \textbf{[10]}$$

The **MME [10] are those for the model**:

$$\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b + \begin{bmatrix} Z_p \\ ½\, Z_n\, P_{np} \end{bmatrix} u_p + \begin{bmatrix} e_p \\ Z_n\,\phi_n + e_n \end{bmatrix} \quad \textbf{[11]}$$

$$E\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b$$

$$\mathrm{var}\begin{bmatrix} u_p \\ \text{--------} \\ e_p \\ Z_n\,\phi_n + e_n \end{bmatrix} = \begin{bmatrix} A_{pp}\alpha & | & 0 & 0 \\ \text{-----} & | & \text{---} & \text{---------} \\ 0 & | & I & 0 \\ 0 & | & 0 & Z_n D_n Z_n{}'\alpha + I \end{bmatrix} \sigma_e^2$$

$$\Rightarrow \quad \text{var}\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} Z_p \\ \tfrac{1}{2} Z_n P_{np} \end{bmatrix} A_{pp}\alpha \begin{bmatrix} Z_p' & \tfrac{1}{2} P_{np}' Z_n' \end{bmatrix} \sigma_e^2 + \begin{bmatrix} I & 0 \\ 0 & Z_n D_n Z_n' \alpha + I \end{bmatrix} \sigma_e^2$$

**The EAM [11] is called the Reduced Animal Model (RAM).**

The **RAM** can also be derived starting from the EAM [6] by expressing the BV of the nonparents in terms of the BV and the Mendelian sampling of their parents, i.e., let:

$$u_n = \tfrac{1}{2} P_{np} u_p + \varphi_n \qquad [12]$$

where

$P_{np}$ = incidence matrix relating nonparents to their known parents,

$\varphi_n$ = vector of Mendelian sampling terms for nonparents, where

$$\phi_{n_i} = \begin{cases} \tfrac{1}{2}\varepsilon_{s_i} + \tfrac{1}{2}\varepsilon_{d_i} & \text{if } s_i \text{ and } d_i \text{ are known} \\ \tfrac{1}{2} u_{d_i} + \tfrac{1}{2}\varepsilon_{s_i} + \tfrac{1}{2}\varepsilon_{d_i} & \text{if } s_i \text{ is known only} \\ \tfrac{1}{2} u_{s_i} + \tfrac{1}{2}\varepsilon_{s_i} + \tfrac{1}{2}\varepsilon_{d_i} & \text{if } d_i \text{ is known only} \\ u_i & \text{if } s_i \text{ and } d_i \text{ are unknown} \end{cases}$$

Substituting $u_n$ in the second set of equations of [6] for expression [12] yields the EAM:

$$\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b + \begin{bmatrix} Z_p & 0 \\ \tfrac{1}{2} Z_n P_{np} & Z_n \end{bmatrix} \begin{bmatrix} u_p \\ \phi_n \end{bmatrix} + \begin{bmatrix} e_p \\ e_n \end{bmatrix} \qquad [13]$$

$$E\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b$$

$$\text{var}\begin{bmatrix} u_p \\ \phi_n \\ --- \\ e_p \\ e_n \end{bmatrix} = \left[\begin{array}{cc|cc} A_{pp}\alpha & 0 & 0 & 0 \\ 0 & D_n\alpha & 0 & 0 \\ ---- & --- & -- & -- \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_n \end{array}\right] \sigma_e^2$$

$$\mathrm{var}\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} Z_p & 0 \\ \tfrac{1}{2} Z_n P_{np} & Z_n \end{bmatrix}\begin{bmatrix} A_{pp}\alpha & 0 \\ 0 & D_n\alpha \end{bmatrix}\begin{bmatrix} Z_p' & \tfrac{1}{2} P_{np}' Z_n' \\ 0 & Z_n' \end{bmatrix}\sigma_e^2 + \begin{bmatrix} I_p & 0 \\ 0 & I_n \end{bmatrix}\sigma_e^2$$

The MME for the **EAM** [13] are:

$$\begin{bmatrix} X_p'X_p + X_n'X_n & X_p'Z_p + \tfrac{1}{2} X_n'Z_n P_{np} & X_n'Z_n \\ Z_p'X_p + \tfrac{1}{2} P_{np}'Z_n'X_n & Z_p'Z_p + A_{pp}^{-1}\alpha^{-1} + \tfrac{1}{4} P_{np}'Z_n'Z_n P_{np} & \tfrac{1}{2} P_{np}'Z_n'Z_n \\ Z_n'X_n & \tfrac{1}{2} Z_n'Z_n P_{np} & Z_n'Z_n + D_n^{-1}\alpha^{-1} \end{bmatrix}\begin{bmatrix} b^\circ \\ \hat{u}_p \\ \phi_n \end{bmatrix} = \begin{bmatrix} X_p'y_p + X_n'y_n \\ Z_p'y_p + \tfrac{1}{2} P_{np}'Z_n'y_n \\ Z_n'y_n \end{bmatrix} \quad [14]$$

Notice that $\boldsymbol{\varphi_n}$ **is uncorrelated to** $\mathbf{u_p}$, thus, **it can be placed together with** $\mathbf{e_n}$ **to form a new residual,** $\dot{\mathbf{e}}_\mathbf{n}$, where

$$\dot{\mathbf{e}}_\mathbf{n} = \mathbf{Z_n\varphi_n} + \mathbf{e_n}$$

**The resulting EAM is the RAM shown in equation [11] above.**

**Remarks:**

(a) If nonparents have **several records** for a trait, the var($\dot{e}$) = var($Z_n\varphi_n + e_n$) is block–diagonal. For instance, if the $i^{th}$ animal has two records:

$$\begin{aligned}
\mathrm{var}\begin{bmatrix} \dot{e}_{i1} \\ \dot{e}_{i2} \end{bmatrix} &= \mathrm{var}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}[\phi_i] + \begin{bmatrix} e_{i1} \\ e_{i2} \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}[d_{ii}\alpha][1\ 1] + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\sigma_e^2 \\
&= \left\{\begin{bmatrix} d_{ii}\alpha + 1 & d_{ii}\alpha \\ d_{ii}\alpha & d_{ii}\alpha + 1 \end{bmatrix}\right\}\sigma_e^2 \\
&= \begin{bmatrix} d_{ii}\sigma_A^2 + \sigma_e^2 & d_{ii}\sigma_A^2 \\ d_{ii}\sigma_A^2 & d_{ii}\sigma_A^2 + \sigma_e^2 \end{bmatrix}
\end{aligned}$$

**If nonparents have only 1 record**, then

$$\mathrm{var}(e) = \mathrm{var}(\varphi_n + e_n)$$

$= $ **diagonal matrix**

Thus, for the i$^{th}$ animal with 1 record

$$\text{var}(\dot{e}) = (d_{ii}\,\alpha + 1)\sigma_e^2$$

$$= (d_{ii}\,\sigma_A^2 + \sigma_e^2)$$

(b) The number of equations for the **RAM** will be fewer than for the other **EAM** which include either $u_n$ or $\varphi_n$, thus they are solved faster than the other **EAM**. If the BLUP of some nonparents were wanted, they could be **backsolved using the formulae developed in remark (a) for the MME in equation [9]**. Also, backsolutions can be obtained using the MME in equation [14] for the **EAM** [13]. From the equations for $\varphi_n$, the BLUP of $\varphi_n$ is:

$$\hat{\varphi}_n = (Z_n'Z_n + D^1\alpha^{-1})^{-1}[Z_n'y_n - Z_n'X_nb^\circ - ½Z_n'Z_nP_{np}\,\hat{u}_p]$$

$$\hat{\varphi}_n = (Z_n'Z_n + D^1\alpha^{-1})^{-1}[Z_n'y_n - Z_n'X_nb^\circ] \qquad \} \text{ data}$$

$$+ (Z_n'Z_n + D^1\alpha^{-1})^{-1}(½Z_n'Z_nP_{np}\,\hat{u}_p] \qquad \} \text{ pedigree}$$

From equation [12], the BLUP of $u_n$ is:

$$\hat{u}_n = ½\, P_{np}\,\hat{u}_p + \hat{\varphi}_n$$

For the i$^{th}$ animal with $n_{\bullet i}$ records:

$$\hat{\phi}_{n_i} = \frac{1}{n_{\bullet i} + \frac{\alpha^{-1}}{d_{ii}}}\left[y_{\bullet i \bullet} - \sum_k n_{ki}\,b_k{}^\circ\right] - \frac{n_{\bullet l}}{n_{\bullet l} + \frac{\alpha^{-1}}{d_{ii}}}\left[½\left(\delta_{s_i}\,\hat{u}_{s_i} + \delta_{d_i}\,\hat{u}_{d_i}\right)\right]$$

$$\hat{\phi}_{n_i} = \frac{d_{ii}}{n_{\bullet i}\,d_{ii} + \alpha^{-1}}\left[y_{\bullet i \bullet} - \sum_k n_{ki}\,b_k{}^\circ\right] - \frac{n_{\bullet l}\,d_{ii}}{n_{\bullet l}\,d_{ii} + \alpha^{-1}}\left[½\left(\delta_{s_i}\,\hat{u}_{s_i} + \delta_{d_i}\,\hat{u}_{d_i}\right)\right]$$

and

$$\hat{u}_{n_i} = ½\,(\delta_{s_i}\,\hat{u}_{s_i} + \delta_{d_i}\,\hat{u}_{d_i}) + \hat{\phi}_{n_i}$$

**Example for the AM and the RAM**

| Animal | Sex | Weaning Weight (kg) | Sire | Dam | Mgs |
|---|---|---|---|---|---|
| 1 | M | | | | |
| 2 | F | | 1 | | |
| 3 | M | 292 | 1 | 2 | 1 |
| 4 | M | 286 | 1 | | |
| 5 | M | 304 | 1 | | |
| 6 | F | 256 | 3 | 2 | 1 |
| 7 | F | 261 | 3 | 6 | 3 |
| 8 | F | 266 | 4 | 7 | 3 |
| 9 | F | 270 | 5 | 8 | 4 |
| 10 | F | 275 | 5 | 9 | 5 |
| 11 | M | 289 | 3 | 6 | 3 |
| 12 | M | 285 | 4 | 7 | 3 |
| 13 | F | 265 | 4 | 8 | 4 |
| 14 | M | 290 | 5 | 9 | 5 |
| 15 | F | 288 | 5 | 10 | 5 |

**Assumptions**

$\sigma_A^{\ 2} \ = \ 22 \text{ kg}^2, \ \sigma_e^{\ 2} \ = \ 88 \text{ kg}^2, \ \sigma_P^{\ 2} \ = \ 110 \text{ kg}^2$

$$\Rightarrow \quad h^2 = \frac{\sigma_A^2}{\sigma_p^2} = 0.2\,; \ \alpha = \frac{\sigma_A^2}{\sigma_e^2} = 0.25 \quad \Rightarrow \quad \alpha^{-1} = 4.0$$

[A] Consider

$$y_{ijk} = \mu + \text{sex}_i + \text{animal}_j + \text{residual}_{ijk}$$

$$E[y_{ijk}] = \mu + \text{sex}_i$$

$$\begin{aligned} \text{var}(y_{ijk}) &= \text{var}(\text{animal}_j) + \text{var}(\text{residual}_{ijk}) \\ &= a_{jj}\,\sigma_A^{\ 2} + \sigma_e^{\ 2} \end{aligned}$$

$$\begin{aligned} \text{cov}(y_{ijk}, y_{i'j'k'}) &= \text{cov}(\text{animal}_j, \text{animal}_{j'}) + \text{cov}(\text{residual}_{ijk}, \text{residual}_{i'j'k'}) \\ &= a_{jj'}\,\sigma_A^{\ 2} + \delta\,\sigma_e^{\ 2}, \text{ where} \end{aligned}$$

$$\delta = \begin{cases} 1 & \text{if } ijk = i'j'k' \quad (\text{diagonal element}) \\ 0 & \text{otherwise} \qquad\quad (\text{offdiagonal element}) \end{cases}$$

In matrix notation, the **AM** is:

$$y = Xb + Zu + e$$

$$E[y] = Xb$$

$$\text{var}\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} A\sigma_A^2 & 0 \\ 0 & I\sigma_e^2 \end{bmatrix} = \begin{bmatrix} A\alpha & 0 \\ 0 & I \end{bmatrix}\sigma_e^2$$

$$= \begin{bmatrix} A(0.25) & 0 \\ 0 & I \end{bmatrix} (88)$$

Explicitly, the vectors and matrices of the **AM** model are:

$$\begin{bmatrix} 292 \\ 286 \\ 304 \\ 256 \\ 261 \\ 266 \\ 270 \\ 275 \\ 289 \\ 285 \\ 265 \\ 290 \\ 288 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 &&&&&&&&&&&& \\ & 1 &&&&&&&&&&& \\ && 1 &&&&&&&&&& \\ &&& 1 &&&&&&&&& \\ &&&& 1 &&&&&&&& \\ &&&&& 1 &&&&&&& \\ &&&&&& 1 &&&&&& \\ &&&&&&& 1 &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& 1 &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& 1 & \\ &&&&&&&&&&&& 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} + e$$

and

$$G = \mathrm{cov}\left\{ \begin{bmatrix} u_3 \\ u_4 \\ \vdots \\ u_{15} \end{bmatrix}, \begin{bmatrix} u_3 & u_4 \cdots u_{15} \end{bmatrix} \right\}.$$

However, to build $A^{-1}$ (and A) directly using Henderson's rules, we need base animals 1 and 2. Thus, instead of **AM**, we will use the following **EAM**:

$$y = Xb + [0 \quad Z]\begin{bmatrix} u_o \\ u_r \end{bmatrix} + e$$

where

$$u_0 = [u_1\ u_2]'$$

$$u_r = [u_3\ u_4\ \ldots\ u_{15}]'$$

$$0 = 13 \times 2 \text{ submatrix of zeroes for } u_0$$

Thus,

$$\begin{bmatrix} 292 \\ 286 \\ 304 \\ 256 \\ 261 \\ 266 \\ 270 \\ 275 \\ 289 \\ 285 \\ 265 \\ 290 \\ 288 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \left[\begin{array}{cc|ccccccccccccc} 0 & 0 & 1 &&&&&&&&&&&& \\ 0 & 0 && 1 &&&&&&&&&&& \\ 0 & 0 &&& 1 &&&&&&&&&& \\ 0 & 0 &&&& 1 &&&&&&&&& \\ 0 & 0 &&&&& 1 &&&&&&&& \\ 0 & 0 &&&&&& 1 &&&&&&& \\ 0 & 0 &&&&&&& 1 &&&&&& \\ 0 & 0 &&&&&&&& 1 &&&&& \\ 0 & 0 &&&&&&&&& 1 &&&& \\ 0 & 0 &&&&&&&&&& 1 &&& \\ 0 & 0 &&&&&&&&&&& 1 && \\ 0 & 0 &&&&&&&&&&&& 1 & \\ 0 & 0 &&&&&&&&&&&&& 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ -- \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} + e$$

$$E[y] = Xb$$

and

$$\mathrm{var}\begin{bmatrix} u_0 \\ u_r \\ e \end{bmatrix} = \begin{bmatrix} A_{00}(0.25) & A_{0r}(0.25) & 0 \\ A_{r0}(0.25) & A_{rr}(0.25) & 0 \\ 0 & 0 & I \end{bmatrix} \quad (88)$$

The inverse of the relationship matrix A is:

$$A^{-1} = (I - ½\, P')\, D^{-1}\, (I - ½\, P)$$

The diagonal elements of A and the elements of D and $D^{-1}$ are:

| i | diagonal of A | diagonal of D | diagonal of $D^{-1}$ |
|---|---|---|---|
| 1 | 1.0 | 1.0 | 1.0 |
| 2 | 1.0 | 0.75 | 1.3333 |
| 3 | 1.25 | 0.5 | 2.0 |
| 4 | 1.0 | 0.75 | 1.3333 |
| 5 | 1.0 | 0.75 | 1.3333 |
| 6 | 1.375 | 0.4375 | 2.2857 |
| 7 | 1.5 | 0.34375 | 2.9091 |
| 8 | 1.171875 | 0.375 | 2.6667 |
| 9 | 1.1484375 | 0.45703125 | 2.1880 |
| 10 | 1.32421875 | 0.462890625 | 2.1603 |
| 11 | 1.5 | 0.34375 | 2.9091 |
| 12 | 1.171875 | 0.375 | 2.6667 |
| 13 | 1.3359375 | 0.45703125 | 2.1880 |
| 14 | 1.32421875 | 0.462890625 | 2.1603 |
| 15 | 1.412109375 | 0.4189453125 | 2.3869 |

The matrix (I – ½ P) is:

$$
\left[\begin{array}{ccccc|ccccc|ccccc}
1 & & & & & & & & & & & & & & \\
-\tfrac{1}{2} & 1 & & & & & & & & & & & & & \\
-\tfrac{1}{2} & -\tfrac{1}{2} & 1 & & & & & & & & & & & & \\
-\tfrac{1}{2} & 0 & 0 & 1 & & & & & & & & & & & \\
-\tfrac{1}{2} & 0 & 0 & 0 & 1 & & & & & & & & & & \\
\hline
0 & -\tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 1 & & & & & & & & & \\
0 & 0 & -\tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} & 1 & & & & & & & & \\
0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} & 1 & & & & & & & \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} & 1 & & & & & & \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & -\tfrac{1}{2} & 1 & & & & & \\
\hline
0 & 0 & -\tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

The $A^{-1}$ matrix, built by Henderson's rules, is:

$$
\left[\begin{array}{cc|ccccccccccccc}
2.5 & -0.167 & -1.0 & -0.667 & -0.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 2.405 & -0.429 & 0 & 0 & -1.143 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & 4.026 & 0 & 0 & 0.312 & -1.455 & 0 & 0 & 0 & -1.455 & 0 & 0 & 0 & 0 \\
 & & & 3.214 & 0 & 0 & 1.333 & -0.786 & 0 & 0 & 0 & -1.333 & -1.094 & 0 & 0 \\
 & & & & 3.557 & 0 & 0 & 0.547 & -0.0138 & -0.483 & 0 & 0 & 0 & -1.080 & -1.194 \\
 & & & & & 3.740 & -1.455 & 0 & 0 & 0 & -1.455 & 0 & 0 & 0 & 0 \\
 & & & & & & 4.242 & -1.333 & 0 & 0 & 0 & -1.333 & 0 & 0 & 0 \\
 & & & & & & & 3.761 & -1.094 & 0 & 0 & 0 & -1.094 & 0 & 0 \\
 & & & & & & & & 3.268 & -1.080 & 0 & 0 & 0 & -1.080 & 0 \\
 & & \text{Symmetric} & & & & & & & 2.757 & 0 & 0 & 0 & 0 & -1.194 \\
 & & & & & & & & & & 2.909 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & 2.667 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 2.188 & 0 & 0 \\
 & & & & & & & & & & & & & 2.160 & 0 \\
 & & & & & & & & & & & & & & 2.387
\end{array}\right]
$$

[16-23]

The MME for the **EAM** are:

$$
\left[\begin{array}{ccccccccccccccccc}
6 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & 7 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & & 10 & -0.667 & -4 & -2.667 & \_2.667 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 9.619 & -1.714 & 0 & 0 & -4.572 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 17.104 & 0 & 0 & 1.247 & -5.818 & 0 & 0 & 0 & -5.818 & 0 & 0 & 0 & 0 \\
 & & & & & 13.855 & 0 & 0 & 5.333 & -3.145 & 0 & 0 & 0 & -5.333 & -4.376 & 0 & 0 \\
 & & & & & & 15.229 & 0 & 0 & 2.188 & -0.0554 & -1.933 & 0 & 0 & 0 & -4.321 & -4.774 \\
 & & & & & & & 15.961 & -5.818 & 0 & 0 & 0 & -5.818 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 17.97 & -5.333 & 0 & 0 & 0 & -5.3330 & 0 & 0 & 0 \\
 & & & & & & & & & 16.043 & -4.375 & 0 & 0 & 0 & -4.376 & 0 & 0 \\
 & & & & & & & & & & 14.073 & -4.321 & 0 & 0 & 0 & -4.321 & 0 \\
 & & & & & \text{Symmetric} & & & & & & 12.028 & 0 & 0 & 0 & 0 & -4.774 \\
 & & & & & & & & & & & & 12.636 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & 11.667 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & 9.752 & 0 & 0 \\
 & & & & & & & & & & & & & & & 9.641 & 0 \\
 & & & & & & & & & & & & & & & & 10.548
\end{array}\right]
\left[\begin{array}{c}
b_1 \\ b_2 \\ \hline u_1 \\ u_2 \\ \hline u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15}
\end{array}\right]
=
\left[\begin{array}{c}
1746 \\ 1881 \\ \hline 0 \\ 0 \\ \hline 292 \\ 286 \\ 304 \\ 256 \\ 261 \\ 266 \\ 270 \\ 275 \\ 289 \\ 285 \\ 265 \\ 290 \\ 288
\end{array}\right]
$$

The vector of solutions for the **EAM** is:

$$\begin{bmatrix} b_1^{\circ} \\ b_2^{\circ} \\ ---- \\ \hat{u}_1 \\ \hat{u}_2 \\ ---- \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ ---- \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \\ \hat{u}_{10} \\ ---- \\ \hat{u}_{11} \\ \hat{u}_{12} \\ \hat{u}_{13} \\ \hat{u}_{14} \\ \hat{u}_{15} \end{bmatrix} = \begin{bmatrix} 291.6375 \\ 269.0400 \\ ---------- \\ -0.4410 \\ -2.0758 \\ ---------- \\ -2.1405 \\ -1.5212 \\ 3.5970 \\ ---------- \\ -3.5008 \\ -3.3839 \\ -2.4499 \\ 0.9258 \\ 3.3714 \\ ---------- \\ -2.8061 \\ -2.8113 \\ -2.1962 \\ 1.8570 \\ 4.9514 \end{bmatrix}$$

The MME for the **AM** (i.e., the **EAM** with $u_0$ absorbed into b and $u_1$) are:

$$
\left[\begin{array}{ccccccccccccccc}
6 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & 7 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & & 15.094 & -1.103 & -1.103 & 0.301 & -5.818 & 0 & 0 & 0 & -5.818 & 0 & 0 & 0 & 0 \\
 & & & 13.140 & -0.714 & -0.085 & 5.333 & -3.145 & 0 & 0 & 0 & -5.33 & -4.3760 & 0 & 0 \\
 & & & & 14.515 & -0.085 & 0 & 2.188 & -0.055 & -1.934 & 0 & 0 & 0 & -4.321 & -4.774 \\
 & & & & & 13.788 & -5.818 & 0 & 0 & 0 & -5.818 & 0 & 0 & 0 & 0 \\
 & & & & & & 17.97 & -5.333 & 0 & 0 & 0 & -5.333 & 0 & 0 & 0 \\
 & & & & & & & 16.043 & -4.376 & 0 & 0 & 0 & -4.376 & 0 & 0 \\
 & & & & & & & & 14.073 & -4.321 & 0 & 0 & 0 & -4.321 & 0 \\
 & & & & & & & & & 12.028 & 0 & 0 & 0 & 0 & -4.774 \\
 & & & & & & & & & & 12.636 & 0 & 0 & 0 & 0 \\
 & & & & & \text{Symmetric} & & & & & & 11.667 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 9.752 & 0 & 0 \\
 & & & & & & & & & & & & & 9.641 & 0 \\
 & & & & & & & & & & & & & & 10.548
\end{array}\right]
\left[\begin{array}{c}
b_1 \\ b_2 \\ \hline u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15}
\end{array}\right]
=
\left[\begin{array}{c}
1746 \\ 1881 \\ \hline 292 \\ 286 \\ 304 \\ 256 \\ 261 \\ 266 \\ 270 \\ 275 \\ 289 \\ 285 \\ 265 \\ 290 \\ 288
\end{array}\right]
$$

The solution vector for the MME for the **AM** (i.e., the **EAM** with $u_0$ absorbed) is:

$$\begin{bmatrix} b_1^o \\ b_2^o \\ \text{----} \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \text{----} \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \\ \hat{u}_{10} \\ \text{----} \\ \hat{u}_{11} \\ \hat{u}_{12} \\ \hat{u}_{13} \\ \hat{u}_{14} \\ \hat{u}_{15} \end{bmatrix} = \begin{bmatrix} 291.6375 \\ 269.040 \\ \text{----------} \\ -2.1405 \\ -1.5212 \\ 3.5970 \\ \text{----------} \\ -3.5008 \\ -3.3839 \\ -2.4499 \\ 0.9258 \\ 3.3714 \\ \text{----------} \\ -2.8061 \\ -2.8113 \\ -2.1962 \\ 1.8570 \\ 4.9514 \end{bmatrix}$$

The BLUP of $u_0$ based on the solution vector of the **AM** are:

$$\hat{u}_0 = A_{0r}\, A_{rr}^{-1}\, \hat{u}_1$$

$$\hat{u}_0 = \begin{bmatrix} 0.4138 & 0.2679 & 0.2679 & 0.0318 & | & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2069 & 0.0186 & 0.0186 & 0.4775 & | & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2.1405 \\ -1.5212 \\ 3.5970 \\ -3.5008 \\ \text{---------} \\ -3.3839 \\ -2.4499 \\ 0.9258 \\ 3.3714 \\ -2.8061 \\ -2.8113 \\ -2.1962 \\ 1.8750 \\ 4.9514 \end{bmatrix}$$

$$\hat{u}_0 = -\left(A^{00}\right)^{-1} A^{0r}\, \hat{u}_1$$

[B] The **RAM** is:

$$\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b + \begin{bmatrix} Z_p \\ \tfrac{1}{2} I_n P_{np} \end{bmatrix} u_p + \begin{bmatrix} e_p \\ I_n \varphi_n + e_n \end{bmatrix}$$

$$E \begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b$$

$$\text{var}\begin{bmatrix} u_p \\ e_p \\ \varphi_n + e_n \end{bmatrix} = \left[\begin{array}{c|cc} A_{pp}(\frac{1}{4}) & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & D_n(\frac{1}{4}) + I \end{array}\right] \quad (88)$$

The diagonals of $A_p$, $D_p$ and $D^{-1}$ are:

| p | $a_{pp}$ | $d_{pp}$ | $d_{pp}^{-1}$ |
|---|---|---|---|
| 1 | 1.0 | 1.0 | 1.0 |
| 2 | 1.0 | 0.75 | 1.3333 |
| 3 | 1.25 | 0.5 | 2.0 |
| 4 | 1.0 | 0.75 | 1.3333 |
| 5 | 1.0 | 0.75 | 1.3333 |
| 6 | 1.375 | 0.4375 | 2.2857 |
| 7 | 1.5 | 0.34375 | 2.9091 |
| 8 | 1.171875 | 0.375 | 2.6667 |
| 9 | 1.1484375 | 0.45703125 | 2.1880 |
| 10 | 1.32421875 | 0.462890625 | 2.1603 |

The $A_{pp}^{-1}$ matrix is:

$$\begin{bmatrix} 2.5 & -0.1667 & -1.0 & -0.6667 & -0.6667 & 0 & 0 & 0 & 0 & 0 \\ & 2.4048 & -0.4286 & 0 & 0 & -1.1429 & 0 & 0 & 0 & 0 \\ & & 3.2987 & 0 & 0 & -0.4156 & -1.4545 & 0 & 0 & 0 \\ & & & 2.0 & 0 & 0 & 0.6667 & -1.333 & 0 & 0 \\ & & & & 2.4204 & 0 & 0 & 0.547 & -0.5539 & -1.0802 \\ & & & & & 3.013 & -1.4545 & 0 & 0 & 0 \\ & & & & & & 3.5758 & -1.3333 & 0 & 0 \\ & & & & & & & 3.2137 & -1.094 & 0 \\ & & & & & & & & 2.7281 & -1.0802 \\ & & & & & & & & & 2.1603 \end{bmatrix}$$

The diagonal elements of $D_n$ and $D^{-1}$ are:

| n | $d_{nn}$ | $d_{nn}^{-1}$ |
|---|---|---|
| 11 | 0.34375 | 2.9091 |
| 12 | 0.375 | 2.6667 |
| 13 | 0.45703125 | 2.1880 |
| 14 | 0.462890625 | 2.1603 |
| 15 | 0.4189453125 | 2.3869 |

The matrix $P_{np}$, which relates parents to progeny, is:

$$P_{np} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the **RAM** looks like:

$$\begin{bmatrix} y_p \\ y_n \end{bmatrix} = \begin{bmatrix} X_p \\ X_n \end{bmatrix} b + \begin{bmatrix} Z_p \\ \tfrac{1}{2} P_{np} \end{bmatrix} u_p + \begin{bmatrix} e_p \\ \phi_n + e_n \end{bmatrix}$$

The matrix $R_2 = D_n\,(1/4) + I$ is:

$$R_2 = \begin{bmatrix} 1.0859375 & & & & \\ & 1.09375 & & & \\ & & 1.11425781 & & \\ & & & 1.11572266 & \\ & & & & 1.10473633 \end{bmatrix}$$

The MME for the **RAM** are:

$$\left[\begin{array}{cc|cccccccccc}
5.731 & 0 & 0 & 0 & 1.460 & 1.457 & 1.448 & 0.460 & 0.457 & 0 & 0.448 & 0 \\
 & 6.803 & 0 & 0 & 0 & 0.449 & 0.4533 & 1.0 & 1.0 & 1.449 & 1.0 & 1.453 \\
\hline
 & & 10.0 & -0.667 & -4.0 & -2.667 & -2.667 & 0 & 0 & 0 & 0 & 0 \\
 & & & 9.619 & -1.714 & 0 & 0 & -4.571 & 0 & 0 & 0 & 0 \\
 & & & & 14.425 & 0 & 0 & -1.432 & -5.818 & 0 & 0 & 0 \\
 & & & & & 9.453 & 0 & 0 & 2.895 & -5.109 & 0 & 0 \\
 & & & & & & 11.32 & 0 & 0 & 2.188 & -1.992 & -4.094 \\
 & & & & & & & 13.282 & -5.818 & 0 & 0 & 0 \\
 & & & & & & & & 15.532 & -5.333 & 0 & 0 \\
 & & & & & & & & & 14.079 & -4.376 & 0 \\
 & & & & & & & & & & 12.137 & -4.321 \\
 & & & & & & & & & & & 9.868
\end{array}\right]
\left[\begin{array}{c} b_1 \\ b_2 \\ \hline u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{array}\right]
=
\left[\begin{array}{c} 1668.622 \\ 1826.511 \\ \hline 0 \\ 0 \\ 425.065 \\ 535.200 \\ 564.308 \\ 389.065 \\ 391.286 \\ 384.913 \\ 399.961 \\ 405.348 \end{array}\right]$$

The vector of solutions for the **RAM** is:

$$\begin{bmatrix} b_1^\circ \\ b_2^\circ \\ ---- \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \\ \hat{u}_7 \\ \hat{u}_8 \\ \hat{u}_9 \\ \hat{u}_{10} \end{bmatrix} = \begin{bmatrix} 291.6375 \\ 269.0403 \\ ------- \\ -0.4410 \\ -2.0758 \\ -2.1405 \\ -1.5212 \\ 3.5970 \\ -3.5008 \\ -3.3839 \\ -2.4499 \\ 0.9258 \\ 3.3714 \end{bmatrix}$$

The vector of deviations of the BV of the nonparents from the midparental BV, i.e., the vector $\varphi_n$, is:

$$\varphi_n = (I + D_n^{-1}\,\alpha^{-1})^{-1}\,[y_n - X_n b_n - ½\,P_{np}\,u_p]$$

$$\varphi_n = W^{-1}[y_n - X_n b_n - ½\,P_{np}\,u_p]$$

where

$$W^{-1} = \text{diag}\,\{w_a^{ii}\}$$

$$W^{-1} = \text{diag}\left\{\frac{d_{n_{ii}}}{d_{n_{ii}} + \alpha^{-1}}\right\}$$

$$W^{-1} = \text{diag}\left\{\begin{matrix} 0.34375 * (0.34375 + 4)^{-1} \\ 0.375 * (0.375 + 4)^{-1} \\ 0.45703125 * (0.45703125 + 4)^{-1} \\ 0.472890625 * (0.462890625 + 4)^{-1} \\ 0.4189453125 * (0.4189453125 + 4)^{-1} \end{matrix}\right\}$$

The BLUP of $\varphi_n$ for the i[th] nonparent is:

$$\hat{\varphi}_{n_i} = w_n^{ii}\left[y_{n_i} - b_{n_i}^o - \tfrac{1}{2}\left(\hat{u}_{s_i} + \hat{u}_{d_i}\right)\right]$$

Thus,

$$\begin{bmatrix} \hat{\varphi}_{11} \\ \hat{\varphi}_{12} \\ \hat{\varphi}_{13} \\ \hat{\varphi}_{14} \\ \hat{\varphi}_{15} \end{bmatrix} = \begin{bmatrix} 0.0791[\,289 - 291.6375 - ½(-2.1405 - 3.5008)] \\ 0.0857[\,285 - 281.6375 - ½(-1.5212 - 3.3839)] \\ 0.1025[\,265 - 269.0403 - ½(-1.5212 - 2.4499)] \\ 0.1037[\,290 - 291.6375 - ½(3.5970 + 0.9258)] \\ 0.0948[\,288 - 269.0403 - ½(3.5970 + 3.3714)] \end{bmatrix}$$

$$= \begin{bmatrix} 0.0144939 \\ -0.3586502 \\ -0.2106119 \\ -0.4043159 \\ 1.4670774 \end{bmatrix}$$

and

$$\hat{u}_{n_i} = ½\left(\hat{u}_{s_i} + \hat{u}_{d_i}\right) + \hat{\phi}_{n_i}$$

$$\begin{bmatrix} \hat{u}_{11} \\ \hat{u}_{12} \\ \hat{u}_{13} \\ \hat{u}_{14} \\ \hat{u}_{15} \end{bmatrix} = \begin{bmatrix} -2.82065 + 0.01449 \\ -2.45255 - 0.35865 \\ -1.98555 - 0.26061 \\ 2.26140 - 0.40432 \\ 3.48420 + 1.45708 \end{bmatrix} = \begin{bmatrix} -2.8061 \\ -2.8113 \\ -2.1962 \\ 1.8570 \\ 4.9514 \end{bmatrix}.$$

**References**

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